MINIMA OF FUNCTIONS OF LINES*

BY

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A function of a line or functional $F[\lambda(x)]$ is a function having as its argument an arc defined over an interval $a \leqq x \leqq b$. It may be regarded as a generalization of a function $F(\lambda_1, \cdots, \lambda_n)$ of a finite number of variables λ_i $(i = 1, \cdots, n)$, the index i with the range $1, 2, \cdots, n$ being replaced by the variable x with the interval $a \leqq x \leqq b$ as its range. It is the purpose of this paper to consider some properties of a functional of this kind which has a minimum value at a particular arc $\lambda_0(x)$. As in the case of a function $F(\lambda_i)$, $(i = 1, \cdots, n)$, this involves the notion of derivatives or differentials of the first and higher orders.

Fréchet has defined first† and second‡ differentials for a function of a line $F(\lambda)$ in terms of which the difference $F(\lambda_0 + \eta) - F(\lambda_0)$ may be expressed, when $\lambda_0(x) + \eta(x)$ is a variation of $\lambda_0(x)$. The first differential is a so-called linear functional $L(\eta)$, and the second is expressible in the form $B(\eta, \eta)$, where $B(u, v)$ is a bilinear functional in the independent arguments $u(x)$, $v(x)$. Riesz has shown§ that a linear functional can always be represented as a Stieltjes integral

$$L(\eta) = \int_a^b \eta(x)\, du(x),$$

and Fréchet has deduced an analogous formula for $B(\eta, \eta)$,

$$B(\eta, \eta) = \int_a^b \int_c^d \eta(x)\eta(y)\, d_{xy}\, p(x, y),$$

in terms of a double integral which is a generalization for two dimensions of the simple Stieltjes form.

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† M. Fréchet, *Sur la notion de différentielle d'une fonction de ligne*, these Transactions, vol. 15 (1914), p. 139. This article will be referred to as Fréchet I.

‡ M. Fréchet, *Sur les fonctions bilinéaires*, these Transactions, vol. 16 (1915), p. 232. This article will be referred to as Fréchet II.

§ F. Riesz, *Sur les opérations fonctionelles linéaires*, Comptes Rendus, 149, p. 974–977; *Démonstration nouvelle d'un théoreme concernant les opérations fonctionelles linéaires*, Annales scientifiques de l'école normale supérieure, vol. 31 (1914). The first article will be referred to as Riesz I; the second, as Riesz II.

The definitions of Fréchet apply only to functionals and differentials having continuity of order zero, which means in the case of $F(\lambda)$ that the difference $F(\lambda_0 + \eta) - F(\lambda_0)$ approaches zero with the maximum of $|\eta(x)|$ on the interval ab. The continuity of $F(\lambda)$ is, on the other hand, of order one when the difference approaches zero only if the maxima of $|\eta(x)|$ and $|\eta'(x)|$ do so simultaneously. The integrals of the calculus of variations have continuity of this latter type, and it was desired to have a theory which should apply to them as a special case. It is shown therefore in the following pages that differentials $L(\eta)$ and $B(\eta, \eta)$ with continuity of order one are expressible in the forms

$$L(\eta) = \int_a^b \eta(x)\,du(x) + \int_a^b \eta'(x)\,du_1(x),$$

$$B(\eta, \eta) = \int_a^b\int_a^b \eta(x)\eta(y)\,d_{xy}\,p(x, y) + 2\int_a^b\int_a^b \eta(x)\eta'(y)\,d_{xy}\,q(x, y) + \int_a^b\int_a^b \eta'(x)\eta'(y)\,d_{xy}\,r(x, y).$$

If the functional $F(\lambda)$ has a minimum at λ_0 then it is proved that u and u_1 must satisfy an equation of the form

$$u_1(x) - \int_a^x u(x)\,dx = kx + l,$$

where k and l are constants. Furthermore, under restrictions which are explained in §§ 4 and 5, a necessary condition for a minimum analogous to the Jacobi condition of the calculus of variations is deduced. It is proved that when $F(\lambda_0)$ is a minimum the equation

$$\int_a^b [u(y)\,d_y\,q(y, x) + u'(y)\,d_y\,r(x, y)] - \int_a^x\int_a^b [u(y)\,d_y\,p(x, y) + u'(y)\,d_y\,q(x, y)]\,dx = kx + l$$

can have no solution $u(x)$, except $u(x) \equiv 0$, vanishing at $x = a$ and a point $x = x'$ between a and b.

These results are deduced in §§ 2 and 5. In §§ 1, 3, and 4 the necessary definitions of differentials are given and their properties discussed. The interpretation of the results of the paper for the integrals of the calculus of variations is given in § 6.

1. Linear functionals and first differentials

If $\mathfrak{L} \equiv [\lambda]$ be a class of arcs in the plane representable in the form

$$y = \lambda(x), \qquad a \leqq x \leqq b,$$

then we mean by the functional operation $F(\lambda)$ a real, single-valued function of the curve λ such that to every λ in the class $\mathfrak{K}$ there corresponds a real number $F(\lambda)$.

If λ_0 is of class $C^{(n)}$,* then by the neighborhood $(\lambda_0)_\delta^n$ of order n is meant the totality of arcs λ of class $C^{(n)}$ satisfying the inequalities:

$$|\lambda(x) - \lambda_0(x)| < \delta, \quad |\lambda'(x) - \lambda_0'(x)| < \delta, \cdots, \quad |\lambda^{(n)}(x) - \lambda_0^{(n)}(x)| < \delta.$$

Consider now a class $\mathfrak{K}$ containing all arcs in a neighborhood $(\lambda_0)_\delta^n$ of an arc λ_0 of class $C^{(n)}$. A functional $F(\lambda)$ is said to have *continuity of order n at λ_0* if for every ϵ there exists a δ such that the inequality

$$|F(\lambda) - F(\lambda_0)| < \epsilon$$

holds whenever λ is in $(\lambda_0)_\delta^n$.

A functional $L(\lambda)$ is said to be linear in a class $\mathfrak{K}$ if for some constant A it has the following properties:†

$$(1) \qquad L(c_1\lambda_1 + c_2\lambda_2) = c_1 L(\lambda_1) + c_2 L(\lambda_2),$$

$$(2) \qquad |L(\lambda)| \leqq A \cdot M(\lambda),$$

whenever λ_1, λ_2 and $c_1\lambda_1 + c_2\lambda_2$, with c_1, c_2 constants, are in $\mathfrak{K}$, and where $M(\lambda)$ denotes the maximum value of $|\lambda|$. In the class $\mathfrak{K}_0$ of arcs λ continuous on $a \leqq x \leqq b$ a functional $L(\lambda)$ which has the property (1) and is continuous with order zero will also have the property (2).‡ F. Riesz§ has shown that such a functional is always in the form

$$L(\lambda) = \int_a^b \lambda(x)\, du(x),$$

where $u(x)$ is of limited variation on the interval ab, and the integral is taken in the sense of Stieltjes.

THEOREM 1. *Let $\mathfrak{K}_1$ be the totality of arcs λ which are of class C' on the interval $a \leqq x \leqq b$. If $L(\lambda)$ has the linear property* (1) *and is continuous with order* 1 *in $\mathfrak{K}$, it is always expressible, indeed in an infinity of ways, in the form*

$$(1) \qquad L(\lambda) = \int_a^b \lambda(x)\, du(x) + \int_a^b \lambda'(x)\, du_1(x),$$

where $u(x)$, $u_1(x)$ are functions of limited variation on $a \leqq x \leqq b$.‖

* An arc $\lambda(x)$ is of class $C^{(n)}$ on $a \leqq x \leqq b$, if $\lambda(x)$, $\lambda'(x)$, $\cdots$, $\lambda^{(n)}(x)$ exist and are continuous on this interval; it is of class $D^{(n)}$ if $\lambda(x)$ is continuous and consists of a finite number of arcs of class $C^{(n)}$, Cf. Bolza, *Lectures on the Calculus of Variations*, p. 7. The arc will be said to be of class D if it is bounded and has a finite number of discontinuities of the first kind.

† Riesz II, p. 10.

‡ F. Riesz I, p. 974.

§ Ibid., pp. 974–977. See also Riesz II, p. 10.

‖ C. A. Fischer, *Note on the order of continuity of functions of lines*, Bulletin of the American Mathematical Society, vol. 23 (1916–7), pp. 88–90.

For, from the hypothesis that λ is of class C', we may write

$$\lambda(x) = \int_a^x \lambda'(x)\,dx + \lambda(a).$$

By the property (1) of a linear functional,

$$L(\lambda) = L\Big(\int_a^x \lambda'(x)\,dx\Big) + L\big(\lambda(a)\big).$$

The first term on the right of the equality is linear and has continuity of order zero in the class of all functions $\lambda'(x)$ which are continuous. Hence with the help of the theorem of Riesz,

$$L(\lambda) = \int_a^b \lambda'(x)\,du_1(x) + \int_a^b \lambda(x)\,du(x),$$

where $u(x)$ is defined by the conditions,

$$u(a) = 0, \qquad u(x) = L(1) \qquad \text{for} \qquad a < x \leqq b.$$

The infinity of ways is evident from the fact that $u(x)$ and $u_1(x)$ may be altered by a constant; but there are even more representations as will be indicated at the end of § 2.

Fréchet* has given a definition of a differential of a functional $F(\lambda)$ defined on the class $\mathfrak{L}_0$ of functions $\lambda(x)$ continuous on $a \leqq x \leqq b$. According to this definition a functional $F(\lambda)$ has a differential at λ_0 if there exists a linear functional $L(\Delta\lambda)$ such that for every arc $\lambda_0 + \Delta\lambda$ in $\mathfrak{L}_0$

$$F(\lambda_0 + \Delta\lambda) - F(\lambda_0) = L(\Delta\lambda) + \epsilon(\Delta\lambda)\cdot M(\Delta\lambda),$$

where $M(\Delta\lambda)$ is the maximum of $|\Delta\lambda|$ on $a \leqq x \leqq b$, and $\epsilon(\Delta\lambda)$ is a functional which approaches zero with $M(\Delta\lambda)$. It is an immediate consequence of Fréchet's definition that $F(\lambda)$ has continuity of order zero at λ_0, a property which is not possessed by the functionals occurring in the calculus of variations. The following definition is however applicable at least to the functionals defined by the integrals of the calculus of variations containing only first derivatives.

DEFINITION 1. *Let λ_0 be an arc of class C' on $a \leqq x \leqq b$ and $F(\lambda)$ a functional defined at least in a neighborhood $(\lambda_0)'_\delta$. Then $F(\lambda)$ is said to have a differential at λ_0 if there exists a linear functional $L(\Delta\lambda)$ with continuity of order one, such that for all arcs $\lambda_0 + \Delta\lambda$ in $(\lambda_0)'_\delta$,*

$$F(\lambda_0 + \Delta\lambda) - F(\lambda_0) = L(\Delta\lambda) + \epsilon(\Delta\lambda)M_1(\Delta\lambda). \tag{2}$$

* Fréchet I, p. 139.

$M_1(\Delta\lambda)$ is the maximum of the values of $|\Delta\lambda|$ and $|\Delta\lambda'|$ on $a \leqq x \leqq b$, and $\epsilon(\Delta\lambda)$ is a functional which vanishes with $M_1(\Delta\lambda)$.

The linear functional $L(\Delta\lambda)$ is always expressible in the form (1), according to Theorem 1.

2. The first variation of $F(\lambda)$

It is proposed in this section to study the properties of the first variation, in other words the first differential, of a functional $F(\lambda)$ which has a minimum or maximum at a particular λ_0. For this purpose we shall be concerned with (1) an arc λ_0 of class C' on the interval $x_1 \leqq x \leqq x_2$, joining the two fixed points (x_1, y_1) and (x_2, y_2); (2) a functional $F(\lambda)$ defined in a neighborhood $(\lambda_0)'_\delta$ of order one of λ_0, which has a differential of the kind described in § 1 at the arc λ_0; (3) the totality $\mathfrak{L}$ of all arcs of class C' joining $(x_1\, y_1)$ with $(x_2\, y_2)$ and lying in $(\lambda_0)'_\delta$.

Definition 2. *The functional $F(\lambda)$ is said to have a minimum at λ_0 in the class $\mathfrak{L}$ if there exists a neighborhood of order one of λ_0 in which $F(\lambda) \geqq F(\lambda_0)$ for every arc λ of $\mathfrak{L}$.*

Consider the special one-parameter family of arcs

$$y = \lambda(x, \alpha) = \lambda_0(x) + \alpha\,\eta(x),$$

for which $\eta(x)$ is of class C' on $x_1 \leqq x \leqq x_2$ and $\eta(x_1) = \eta(x_2) = 0$. These will all be in $\mathfrak{L}$ for sufficiently small values of α, and the value of $F(\lambda)$ on any one of them is from formula (2)

$$F(\lambda(x, \alpha)) = F(\lambda_0) + \alpha L(\eta) + \alpha M(\eta)\,\epsilon(\alpha\eta).$$

Then will follow readily

Lemma 1. *If $F(\lambda_0)$ is a minimum according to the definition given above, then*

$$L(\eta) = \int_{x_1}^{x_2} \eta\, du + \int_{x_1}^{x_2} \eta'\, du_1 \tag{3}$$

must vanish for every function $\eta(x)$ of class C' on $x_1\, x_2$ such that

$$\eta(x_1) = \eta(x_2) = 0.$$

We shall next determine a necessary and sufficient condition that the sum of two integrals of form (3) shall vanish as described. Since the value of a Stieltjes integral is unaltered if the function of limited variation is changed at a finite or denumerable infinity of points between x_1 and x_2, and since the discontinuities of a function of limited variation are denumerable, we may take $u(x)$, $u_1(x)$ to be regular* for $x_1 < x < x_2$; that is, at every point between x_1 and x_2, $2\,u(x) = u(x + 0) + u(x - 0)$.

* Fréchet II, p. 217.

According to a well-known property of a Stieltjes integral,*

$$(4)\qquad \int_{x_1}^{x_2} \eta(x)\,du(x) = \eta(x_2)\,u(x_2) - \eta(x_1)\,u(x_1) - \int_{x_1}^{x_2} u(x)\,d\eta(x),$$

and since $\eta(x)$ is of class C' it follows readily from the definition of a Stieltjes integral that the last integral is also expressible as an ordinary Riemann integral

$$\int_{x_1}^{x_2} u(x)\,\eta'(x)\,dx.$$

Furthermore one may prove without difficulty the relation

$$\int_{x_1}^{x_2} \eta'(x)\,u(x)\,dx = \int_{x_1}^{x_2} \eta'(x)\,d\int_{x_1}^{x} u(x)\,dx,$$

so that with the help of the values $\eta(x_1) = \eta(x_2) = 0$ the expression for $L(\eta)$ from (3) may be written in the form

$$(5)\qquad \int_{x_1}^{x_2} [\,\eta(x)\,du(x) + \eta'(x)\,du_1(x)\,] = \int_{x_1}^{x_2} \eta'(x)\,d\hat{u}(x),$$

where

$$\hat{u}(x) = u_1(x) - \int_{x_1}^{x} u(x)\,dx.$$

Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be any four points of the interval $x_1 x_2$ such that

$$x_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < x_2, \qquad \beta_1 - \alpha_1 = \beta_2 - \alpha_2.$$

Let h be a positive number such that

$$x_1 < \alpha_1 - h < \beta_1 + h < \alpha_2 - h < \beta_2 + h < x_2.$$

Define a continuous function $\eta'(x)$ by the conditions

$\eta'(x) \equiv 0$ for $x_1 \leqq x \leqq \alpha_1 - h,\ \beta_1 + h \leqq x \leqq \alpha_2 - h,\ \beta_2 + h \leqq x \leqq x_2,$

$\eta'(x) \equiv 1$ for $\alpha_1 \leqq x \leqq \beta_1,$

$\eta'(x) \equiv -1$ for $\alpha_2 \leqq x \leqq \beta_2,$

and the condition that $\eta'(x)$ is linear in the remaining parts of the interval $x_1 x_2$. Then the function

$$\eta(x) = \int_{x_1}^{x} \eta'(x)\,dx$$

is of class C' and has $\eta(x_1) = \eta(x_2) = 0$.

* T.-J. Stieltjes, *Récherches sur les fractions continues*, Annales de la Faculté des Sciences de Toulouse, vol. VIII (1894), J. 72.

The substitution of $\eta'(x)$ so defined in the expression (4) gives

$$\frac{1}{h}\int_{\alpha_1-h}^{\alpha_1}(x-\alpha_1+h)\,d\hat{u}+\int_{\alpha_1}^{\beta_1}d\hat{u}-\frac{1}{h}\int_{\beta_1}^{\beta_1+h}(x-\beta_1-h)\,d\hat{u}$$
$$-\frac{1}{h}\int_{\alpha_2-h}^{\alpha_1}(x-\alpha_2+h)\,d\hat{u}-\int_{\alpha_2}^{\beta_2}d\hat{u}+\frac{1}{h}\int_{\beta_2}^{\beta_2+h}(x-\beta_2-h)\,d\hat{u},$$

and this must vanish for every choice of $\alpha_1, \beta_1, \alpha_2, \beta_2, h$ satisfying the conditions described above if $F(\lambda)$ is to have a minimum at λ_0. The transformation (4) applied to this expression gives, after some simplification, the necessary condition for a minimum

$$\frac{1}{h}\int_{\beta_1}^{\beta_1+h}\hat{u}(x)\,dx-\frac{1}{h}\int_{\alpha_1-h}^{\alpha_1}\hat{u}(x)\,dx=\frac{1}{h}\int_{\beta_2}^{\beta_2+h}\hat{u}(x)\,dx-\frac{1}{h}\int_{\alpha_2-h}^{\alpha_2}\hat{u}(x)\,dx.$$

If we apply the mean value theorem to each of the above integrals and take the limit as h tends to zero we obtain

$$\hat{u}(\beta_1+0)-\hat{u}(\alpha_1-0)=\hat{u}(\beta_2+0)-\hat{u}(\alpha_2-0), \tag{6}$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2$ satisfy the restrictions described above.

By the definition of $\hat{u}(x)$ its discontinuities are identical with those of $u_1(x)$ and are therefore denumerable. Select α_1 arbitrarily between x_1, and x_2 and let $\alpha_2 > \alpha_1$ be a value at which $\hat{u}(x)$ is continuous. If in the equation (6) we let β_1 and β_2 approach α_1 and α_2 respectively, it follows that

$$\hat{u}(\alpha_1+0)-\hat{u}(\alpha_1-0)=\hat{u}(\alpha_2+0)-\hat{u}(\alpha_2-0).$$

The second member of this equality is zero by the hypothesis that $\hat{u}(x)$ is continuous at α_2. Hence

$$\hat{u}(\alpha_1+0)-\hat{u}(\alpha_1-0)=0,$$

and the same relation holds for $u_1(x)$ whose discontinuities are coincident with those of $\hat{u}(x)$. Since α_1 was an arbitrarily selected point between x_1 and x_2, and since $u_1(x)$ was taken to be regular, it follows from the above that *$u_1(x)$ and also $\hat{u}(x)$ are continuous everywhere between x_1 and x_2, so that equation* (6) *may be written*

$$\hat{u}(\beta_1)-\hat{u}(\alpha_1)=\hat{u}(\beta_2)-\hat{u}(\alpha_2).$$

The function $\hat{u}(x)$, being continuous and of limited variation for $x_1 < x < x_2$ has a derivative everywhere except on a set of points of measure zero.* Let α_1 be an arbitrarily selected point on $x_1 x_2$ and let $\alpha_2 > \alpha_1$ be a point where $\hat{u}(x)$ has a well defined derivative. The differences $\beta_1-\alpha_1$ and $\beta_2-\alpha_2$ being equal we may write

* Vallée Poussin, *Cours d'analyse infinitésimale*, 3d edition, vol. I, p. 275.

$$\frac{\hat{u}(\beta_1) - \hat{u}(\alpha_1)}{\beta_1 - \alpha_1} = \frac{\hat{u}(\beta_2) - \hat{u}(\alpha_2)}{\beta_2 - \alpha_2}. \tag{7}$$

If we let $\beta_1 \to \alpha_1$, $\beta_2 \to \alpha_2$ in such a manner that $\beta_2 - \alpha_2$ and $\beta_1 - \alpha_1$ remain equal, the right member of (7) tends to a limit and the left member therefore approaches the same limit. Hence we have the result that $\hat{u}(x)$ has a derivative everywhere between x_1 and x_2 and that this derivative has a constant value,

$$\hat{u}'(x) = k, \qquad x_1 < x < x_2.$$

We are now able to prove that $\hat{u}(x)$, and therefore $u_1(x)$, is continuous also at the points x_1 and x_2. To do this, consider the equation

$$\int_{x_1}^{x_2} \eta'(x)\,d\hat{u}(x) = 0.$$

Since $\hat{u}(x)$ is discontinuous at most at x_1 and x_2 and is linear between these values, this equation may be written

$$\eta'(x_1)[\hat{u}(x_1+0) - \hat{u}(x_1)] + \eta'(x_2)[\hat{u}(x_2) - \hat{u}(x_2-0)] + k\int_{x_1}^{x_2} \eta'(x)\,dx = 0,$$

or, since $\eta(x_1) = \eta(x_2) = 0$,

$$\eta'(x_1)[\hat{u}(x_1+0) - \hat{u}(x_1)] + \eta'(x_2)[\hat{u}(x_2) - \hat{u}(x_2-0)] = 0,$$

a result which must be true for every $\eta(x)$ of class C' vanishing at x_1 and x_2. Hence it follows that

$$\hat{u}(x_1+0) = \hat{u}(x_1), \qquad \hat{u}(x_2-0) = \hat{u}(x_2),$$

and that the same relation holds for $u_1(x)$. We have accordingly proved

LEMMA 2. *If $u(x)$, $u_1(x)$ are of limited variation on the interval x_1x_2 and regular for $x_1 < x < x_2$, and if the integral*

$$\int_{x_1}^{x_2} [\eta(x)\,du(x) + \eta'(x)\,du_1(x)] \tag{8}$$

vanishes for every $\eta(x)$ of class C' such that $\eta(x_1) = \eta(x_2) = 0$, then a relation of the form

$$\hat{u}(x) \equiv u_1(x) - \int_{x_1}^{x} u(x)\,dx = kx + l \tag{9}$$

must hold everywhere on the interval x_1x_2, k and l being constants.

Conversely, if the last equation is true the integral (8) *vanishes for all functions $\eta(x)$ with the properties just described.*

The converse follows readily by substituting $\hat{u}(x) = kx + l$ in the formula (5).

It is interesting also to find the additional conditions on $u(x)$ and $u_1(x)$ which must hold if the expression (3) for the functional $L(\eta)$ is to vanish for all functions $\eta(x)$ of class C' on the interval $x_1 x_2$, whether or not the conditions $\eta(x_1) = \eta(x_2) = 0$ are satisfied. The necessary condition of the lemma just proved must be satisfied in this case also, and the relation analogous to (5) now is

$$\int_{x_1}^{x_2} [\eta(x)\,du(x) + \eta'(x)\,du_1(x)] = [\eta(x)u(x)]_{x_1}^{x_2} + \int_{x_1}^{x_2} \eta'(x)\,d\hat{u}(x)$$
$$= [\eta(x)u(x)]_{x_1}^{x_2} + k\int_{x_1}^{x_2} \eta'(x)\,dx = \eta(x_2)[u(x_2)+k] - \eta(x_1)[u(x_1)+k].$$

This last expression vanishes for every $\eta(x)$ of class C'; hence we conclude that

$$k = -u(x_2) = -u(x_1),$$

and it follows from the relation (9) that

$$\begin{aligned} u_1'(x_1) &= u(x_1+0) - u(x_1), \\ u_1'(x_2) &= u(x_2-0) - u(x_2). \end{aligned} \tag{10}$$

By the preceding arguments we have therefore arrived at the following theorem:

THEOREM 2. *If $F(\lambda_0)$ be a maximum or minimum according to the conditions described in Definition 2, then the functions $u(x)$ and $u_1(x)$ occurring in the first variation*

$$\int_{x_1}^{x_2} [\eta(x)\,du(x) + \eta'(x)\,du_1(x)]$$

must satisfy the relation

$$u_1(x) - \int_{x_1}^{x} u(x)\,dx = kx + l.$$

If $F(\lambda_0)$ be a maximum or minimum with respect to the values of $F(\lambda)$ for all arcs whatsoever of class C' on $x_1 \leqq x \leqq x_2$ lying in a neighborhood $(\lambda_0)'$, then the additional conditions

$$u_1'(x_1) = u(x_1+0) - u(x_1),$$
$$u_1'(x_2) = u(x_2-0) - u(x_2)$$

must be satisfied.

In relations (9) and (10) we have the conditions that integral (8) shall vanish for all η's of class C' on $x_1 x_2$. Consequently, if we have two representations of the same linear functional,

$$\int_{x_1}^{x_2} [\eta\,du + \eta'\,du_1] \qquad \text{and} \qquad \int_{x_1}^{x_2} [\eta\,dv + \eta'\,dv_1],$$

then $u - v$ and $u_1 - v_1$ must satisfy the conditions that

$$\int_{x_1}^{x_2} [\eta\, d(u - v) + \eta'\, d(u_1 - v_1)]$$

shall vanish for all η's of class C' on $x_1 x_2$.

3. Bilinear functionals and second differentials

For the purpose of discussing further the conditions which characterize a minimum of the functional $F(\lambda)$ we next introduce the notion of a second differential. The basis of the investigation is found in the paper of Fréchet on bilinear functionals.*

If $\lambda(x)$, $\mu(y)$ are two continuous functions defined on the two intervals $a \leqq x \leqq b$, $c \leqq y \leqq d$, then according to Fréchet, $B(\lambda, \mu)$ is a bilinear functional of λ, μ if it is a linear functional of λ for a fixed μ and a linear functional of μ for a fixed λ. The bilinear functional $B(\lambda, \mu)$ so defined has continuity of order zero simultaneously in λ, μ and it has also the property that there is a constant P such that

$$|B(\lambda, \mu)| \leqq PM(\lambda) M(\mu).$$

By use of the theorem of Riesz on linear functionals Fréchet derives a representation of a bilinear functional by means of two iterated Stieltjes integrals. He then shows that each of these iterated integrals is equal to a double integral of the form

$$B(\lambda, \mu) = \int_a^b \int_c^d \lambda(x)\mu(y)\, d_{xy}\, p(x, y)$$

$$= \lim_{\delta \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda(x_i')\mu(y_j')\Delta_{ij}\, p(x, y),$$

where the values x_i $(i = 1, \cdots, m)$ are the division points of a partition of the interval $a \leqq x \leqq b$ of norm δ, the y_j $(j = 1, \cdots, n)$ are division points for a similar partition of $c \leqq y \leqq d$, and where $\Delta_{ij}\, p(x, y)$ is the second difference

$$\Delta_{ij}\, p(x, y) = p(x_i y_j) - p(x_{i-1}, y_j) - p(x_i, y_{j-1}) + p(x_{i-1}, y_{j-1}).$$

The function $p(x, y)$ is of limited variation with respect to the set (x, y) and with respect to x and y separately, the total variation with respect to the set (x, y) being defined as the upper bound of the expression

$$\Big|\sum_i \sum_j \epsilon_i \epsilon_j' \Delta_{ij}\, p(x, y)\Big|,$$

where ϵ_i, ϵ_j are of arbitrary signs but in absolute value equal to unity.

* Fréchet II, pp. 215–234.

THEOREM 3. *If λ, μ are of class C' on the intervals ab, cd and if $B(\lambda, \mu)$ is linear and continuous with order one in each variable when the other is fixed, then $B(\lambda, \mu)$ is expressible in the form*

$$B(\lambda, \mu) = \int_a^b \int_c^d \lambda(x)\mu(y)\, d_{xy}\, p(x, y) + \int_a^b \int_c^d \lambda'(x)\mu(y)\, d_{xy}\, q'(x, y)$$

$$+ \int_a^b \int_c^d \lambda(x)\mu'(y)\, d_{xy}\, q''(x, y) + \int_a^b \int_c^d \lambda'(x)\mu'(y)\, d_{xy}\, r(x, y),$$

where the functions p, q', q'', r are of limited variation with respect to x, y together and with respect to each separately.

By hypothesis λ and μ are expressible in the form

$$\lambda(x) = \int_a^x \lambda'(x)\, dx + \lambda(a), \qquad \mu(y) = \int_c^y \mu'(y)\, dy + \mu(c).$$

Then by the properties of $B(\lambda, \mu)$,

$$B(\lambda, \mu) = B\left(\int_a^x \lambda'(x)\, dx, \int_c^y \mu'(y)\, dy\right) + B\left(\int_a^x \lambda'(x)\, dx, \mu(c)\right)$$

$$+ B\left(\lambda(a), \int_c^y \mu'(y)\, dy\right) + B(\lambda(a), \mu(c)).$$

Since the first functional in the right member of the equality is linear and has continuity of order zero in λ', μ', the second in λ', μ, the third in λ, μ', the fourth in λ, μ, we have by help of the Fréchet theorem the desired form for $B(\lambda, \mu)$.

A definition of a second differential will now be given which is somewhat different from that of Fréchet.* Consider an arc $\lambda_0(x)$ which is of class C' on the interval $x_1 \leqq x \leqq x_2$, and let $F(\lambda)$ be a functional which is well defined for all arcs λ of class C' on $x_1 x_2$ and lying in a neighborhood $(\lambda_0)'_\delta$ of order one of λ_0.

DEFINITION 3. *The functional $F(\lambda)$ will be said to have a second differential at λ_0 if there exist a linear functional $L(\lambda)$ and a bilinear functional $B(\lambda, \mu)$ having continuity of order one for all arcs λ, μ of class C' on $x_1 x_2$ such that*

$$F(\lambda_0 + \eta) = F(\lambda_0) + L(\eta) + B(\eta, \eta) + M^2(\eta)\epsilon(\eta)$$

for every arc $\lambda_0 + \eta$ of class C' in a neighborhood of order one of λ_0. The symbol $M(\eta)$ represents the maximum of $|\eta|$, $|\eta'|$ on the interval $x_1 x_2$ and $\epsilon(\eta)$ is a functional which vanishes with $M(\eta)$.

If we consider in particular a family of arcs of the form

$$y = \lambda_0(x) + \alpha\eta(x)$$

* Fréchet II, p. 232.

where $\eta(x)$ is of class C' on the interval $x_1 x_2$ and $\eta(x_1) = \eta(x_2) = 0$, then

$$F(\lambda_0 + \alpha\eta) - F(\lambda_0) = \alpha L(\eta) + \alpha^2 B(\eta, \eta) + \alpha^2 M^2(\eta)\epsilon(\eta),$$

and the following lemma can readily be proved.

LEMMA 3. *If $F(\lambda_0)$ is a minimum in the sense described in Definition 2, then the conditions*

$$L(\eta) = 0, \qquad B(\eta, \eta) \geqq 0$$

must hold for every function $\eta(x)$ of class C' on the interval $x_1 x_2$ having

$$\eta(x_1) = \eta(x_2) = 0.$$

4. Theorems concerning integrals of Stieltjes and Fréchet

The theorems of the preceding sections on linear and bilinear functionals presuppose that the arcs considered are continuous. In the study of the second variation which follows it has been found convenient to introduce arcs which have a finite number of discontinuities of the first kind on the interval $x_1 x_2$ and to consider linear and bilinear functionals defined for such arcs and expressible as integrals of Stieltjes or of Fréchet. For this purpose it will be necessary to show the existence of the integrals under the new hypothesis and to prove the validity of certain relations, some of which have already been proved for the original definitions of the integrals.

THEOREM 4. *If $\alpha(x)$ is continuous and of limited variation on the interval ab, and $f(x)$ is of class D, then the integral*

$$\int_a^b f(x)\, d\alpha(x)$$

exists.

This is a special case of a theorem of G. A. Bliss.*

THEOREM 5. *If $\alpha(x)$ is continuous and of limited variation on ab and $f(x)$ is of class D, then the integral*

$$\int_a^b \alpha(x)\, df(x)$$

exists, and the relation

* *A necessary and sufficient condition for the existence of a Stieltjes integral*, Proceedings of the National Academy of Sciences, vol. 3, pp. 633–637, November, 1917.

W. H. Young states and proves this theorem for a monotonic function $\alpha(x)$ in his paper *On integration with respect to a function of bounded variation*, Proceedings of the London Mathematical Society, ser. 2, vol. 13 (1914), p. 133. Young states but does not prove the theorem for a double integral for an arbitrary function of 2 variables of limited variation in the paper, *On multiple integration by parts and a second theorem of the mean*, ibid., vol. 16 (1917).

$$\int_a^b f(x)\,d\alpha(x) + \int_a^b \alpha(x)\,df(x) = f(b)\alpha(b) - f(a)\alpha(a)$$

*holds.**

THEOREM 6. *If $f(x)$ is of class D' on ab, and*

$$A(x) = \int_a^x \alpha(x)\,dx + A(a),$$

where $\alpha(x)$ is of limited variation on the same interval, then

$$\int_a^b f'(x)\,dA(x) = [f(x)\alpha(x)]_a^b - \int_a^b f(x)\,d\alpha(x).$$

By definition,

$$\int_a^b f'(x)\,dA(x) = \lim \sum_k f'(\xi_k)[A(x_k) - A(x_{k-1})].$$

If the discontinuities of $f'(x)$ are among the division points of the interval ab, and the points ξ_k are properly chosen, the right member of the above equation may be written in the form,

$$\lim \sum_k [f(x_k) - f(x_{k-1})] \frac{A(x_k) - A(x_{k-1})}{x_k - x_{k-1}}$$
$$= \lim \sum_k [f(x_k) - f(x_{k-1})] \cdot \alpha_k,$$

where α_k is a value between the maximum and minimum of $\alpha(x)$ on the interval $x_{k-1}x_k$. It is easily shown that the expressions

$$\sum_k \alpha_k [f(x_k) - f(x_{k-1})], \qquad \sum_k \alpha(\xi_k)[f(x_k) - f(x_{k-1})],$$

in which $\alpha(\xi_k)$ is the value of $\alpha(x)$ at an arbitrary point in the interval $x_{k-1}x_k$, and α_k a number which lies between the maximum and minimum of $\alpha(x)$ on the interval, both approach the same limit, so that

$$\int_a^b f'(x)\,dA(x) = \int_a^b \alpha(x)\,df(x).$$

From this last equality and the relation of Theorem 5 follows the desired relation of the theorem.

Consider a function $p(x, y)$ which is

(1) continuous on the square $S : a \leqq x \leqq b,\ a \leqq y \leqq b$;

(2) expressible in the form

* For a proof of this theorem see G. A. Bliss, *Integrals of Lebesgue*, Bulletin of the American Mathematical Society, 2d series, vol. 24, No. 1, pp. 1–47; H. E. Bray, *Elementary properties of the Stieltjes integral*, Annals of Mathematics, vol. 20 (1919), p. 185.

$$p(x, y) = \int_a^x p_1(x, y)\,dx + p(a, y) = \int_a^y p_2(x, y)\,dy + p(x, a),$$

where $p_1(x, y)$ and $p_2(x, y)$ are both of limited variation in y uniformly with respect to x, and in x uniformly with resepct to y.

The function $p_2(x, y)$, for example, is of limited variation in x uniformly with respect to y when there is a constant V such that the total variation of $p_2(x, y)$ with respect to x on the interval ab is less than V for every y on ab.*

Consider a subdivision of S into rectangles by abscissas x_i $(i = 0, 1, \cdots, m)$ with $x_0 = a$, $x_m = b$, and ordinates y_j $(j = 0, 1, \cdots, n)$ with $y_0 = a$, $y_n = b$, and let $\Delta_{ij}\, p(x, y)$ represent as before the second difference

$$\Delta_{ij}\, p(x, y) = p(x_i y_j) - p(x_i y_{j-1}) - p(x_{i-1}, y_j) + p(x_{i-1}, y_{j-1}).$$

LEMMA 4. *The function p is of limited variation in the sense that the sums $\sum_{i=1}^m \sum_{j=1}^n |\Delta_{ij}\, p|$ have a common upper bound P for all subdivisions of S of the type described above.*

If the lemma is true the function p is evidently of limited variation in the weaker sense of Fréchet also.

To prove the lemma write

$$\sum_i \sum_j |\Delta_{ij}\, p| \leqq \sum_j \int_{y_{j-1}}^{y_j} \sum_i |p_2(x_i, y) - p_2(x_{i-1}, y)|\,dy \leqq V(b-a).$$

THEOREM 7. *If $\lambda(x)$, $\mu(x)$ are of class D on $a \leqq x \leqq b$ the integral*

$$\int\int_S \lambda(x)\mu(y)\,d_{xy}\, p(x, y) \tag{11}$$

is well defined in the sense of Frechet.

Consider a sum

$$\sigma = \sum_{ij} \lambda(\xi_i)\mu(\eta_j)\Delta_{ij}\, p,$$

where ξ_i and η_j are arbitrarily chosen values in the intervals $x_{i-1}\, x_i$ and $y_{j-1}\, y_j$ respectively. The sum of the terms of σ belonging to a single row of rectangles has absolute value less than

$$\sum_i M^2 \int_{y_{j-1}}^{y_j} |p_2(x_i, y) - p_2(x_{i-1}, y)|\,dy \leqq M^2 V(y_j - y_{j-1}),$$

where M is the maximum of the values of $|\lambda(x)|$ and $|\mu(x)|$ on ab, and the sum of the terms of several rows is therefore in absolute value $\leqq M^2 Vw$, where w is the sum of the width of the rows. The same property is true for columns.

* H. E. Bray, loc. cit., p. 180.

Let r and s be the number of discontinuities of $\lambda(x)$ and $\mu(y)$ on ab respectively. Then the discontinuities of the product $\lambda(x)\mu(y)$ occur on r lines parallel to the y-axis and on s lines parallel to the x-axis. If the intervals $x_{i-1}x_i$ and $y_{j-1}y_j$ all have lengths $\leqq 2\delta$, the sum of the terms of σ corresponding to rectangles containing points of these lines of discontinuity will have absolute value not exceeding $(r+s)M^2V2\delta$. Furthermore, since $\lambda(x)$ and $\mu(y)$ are of class D on ab, the norm δ can be chosen so small that the oscillation of the product $\lambda(x)\mu(y)$ in the rectangles containing none of its discontinuities will be less than ϵ.

Consider now a sum σ' formed by subdividing the intervals $x_{i-1}x_i$ and $y_{j-1}y_j$ which were used to form σ. Then

$$|\sigma - \sigma'| \leqq \epsilon P + 4(r+s)M^2V\delta .$$

For the first term on the right exceeds the absolute value of the difference of the parts of σ and σ' belonging to rectangles containing no discontinuities of $\lambda(x)\mu(y)$; and one half the second term exceeds the absolute value of the sum of the remaining terms of either σ or σ'.

If two sums σ and σ'' of norm δ are given, a third sum σ' can be formed by using all of their division points, and its rectangles will be subdivisions of those of σ'' as well as those of σ. From the preceding paragraph it follows that the difference

$$|\sigma - \sigma''| \leqq |\sigma - \sigma'| + |\sigma' - \sigma''|$$

can be made less than ϵ' by taking the norm δ sufficiently small. It follows readily then by the usual arguments that the limit of σ as δ approaches zero exists.

THEOREM 8. *If $\lambda(x)$ and $\mu(x)$ are of class D on the interval ab the two integrals*

$$(12) \qquad \int_a^b \lambda(x)\,d_x \int_a^b \mu(y)\,d_y\,p(x,y), \qquad \int_a^b \mu(y)\,d_y \int_a^b \lambda(x)\,d_x\,p(x,y)$$

exist and are equal to the integral (11) *of Theorem* 7.

Since $p(x,y)$ is continuous and of limited variation in y, the integral

$$\phi(x) = \int_a^b \mu(y)\,d_y\,p(x,y)$$

surely exists for every value of x.* It is to be proved first that it is also a continuous function of x. For this purpose, let the discontinuities of $\mu(y)$ on $a \leqq y \leqq b$ be enclosed in a set of intervals $\alpha_k\beta_k$ $(k = 1, \cdots, s)$ of total

* See Bliss, *A necessary and sufficient condition for the existence of a Stieltjes integral*, Proceedings of the National Academy of Sciences, vol. 3, pp. 633–637, November, 1917.

length less than ϵ, and let $\beta_0 = a$, $\alpha_{s+1} = b$. Then

$$\phi(x) = \sum_{k=1}^{s+1} \int_{\beta_{k-1}}^{\alpha_k} \mu(y)\, d_y\, p(x, y) + \sum_{k=1}^{s} \int_{\alpha_k}^{\beta_k} \mu(y)\, d_y\, p(x, y). \tag{13}$$

Each term of the first sum is a continuous function of x according to a theorem of Bray,* and the second sum has an absolute value less than $MN\epsilon$, where N is the maximum of $|p_2(x, y)|$ in S. For we have the relations

$$|p(x, y_j) - p(x, y_{j-1})| = \left| \int_{y_{j-1}}^{y_j} p_2(x, y)\, dy \right| \leqq N(y_j - y_{j-1}),$$

and hence

$$\sum_{k=1}^{s} \left| \int_{\alpha_k}^{\beta_k} \mu(y)\, d_y\, p(x, y) \right| \leqq \sum_{k=1}^{s} MN(\beta_k - \alpha_k) < MN\epsilon.$$

The continuity of $\phi(x)$ follows readily from the properties of the sums in the expression (13).

The function $\phi(x)$ is also of limited variation,† so that by the theorem of Bliss cited above the integrals (12) exist.

It remains to show that the integrals (12) are equal to (11). For a sufficiently fine x-partition,

$$\left| \int_a^b \lambda(x)\, d\phi(x) - \sum_i \lambda(\xi_i)[\phi(x_i) - \phi(x_{i-1})] \right| < \epsilon/2. \tag{14}$$

But

$$\sum_{i=1}^{n} \lambda(\xi_i)[\phi(x_i) - \phi(x_{i-1})] - \int_a^b \mu(y)\, d\rho(y) = 0, \tag{15}$$

where

$$\rho = \sum_{i=1}^{n} \lambda(\xi_i)[p(x_i, y) - p(x_{i-1}, y)]$$

is continuous and of limited variation. The integral in (15) exists and for a sufficiently fine y-partition

$$\left| \int_a^b \mu(y)\, d\rho(y) - \sum_{j=1}^{s} \mu(\eta_j)[\rho(y_j) - \rho(y_{j-1})] \right| < \epsilon/2. \tag{16}$$

By adding (14), (15), (16), and writing ρ out in full, we have

$$\left| \int_a^b \lambda(x)\, d\phi(x) - \sum_{ij} \lambda(\xi_i)\mu(\eta_j)\Delta_{ij}\, p(x, y) \right| < \epsilon,$$

which proves the theorem.

THEOREM 9. *If $p(x, y)$ is of limited variation in y uniformly with respect to x and $\eta(y)$ is continuous, then*

* Loc. cit., p. 180.

† Fréchet II, p. 229.

$$\int_a^x \int_a^b \eta(y)\, d_y\, p(x, y)\, dx = \int_a^b \eta(y)\, d_y \int_a^x p(x, y)\, dx.$$

If ω represents the maximum oscillation of $\eta(y)$ on the intervals $y_{j-1} y_j$, then

$$\left| \int_a^b \eta(y)\, d_y\, p(x, y) - \sum_{j=1}^n \eta(y_j')\,[\, p(x, y_j) - p(x, y_{j-1})\,] \right| \leqq \omega V.^*$$

Consider the relation

$$\left| \int_a^x \int_a^b \eta(y)\, d_y\, p(x, y)\, dx - \int_a^x \sum_{j=1}^n \eta(y_j')\,[\, p(x, y_j) - p(x, y_{j-1})\,]\, dx \right|$$
$$= \left| \int_a^x \left\{ \int_a^b \eta(y)\, d_y\, p(x, y) - \sum_{j=1}^n \eta(y_j')\,[\, p(x, y_j) - p(x, y_{j-1})\,] \right\} dx \right|$$
$$\leqq \int_a^x \left| \int_a^b \eta(y)\, d_y\, p(x, y) \right.$$
$$\left. - \sum_{j=1}^n \eta(y_j')\,[\, p(x, y_j) - p(x, y_{j-1})\,] \right| dx \leqq \omega V(b-a) \leqq \epsilon/2,$$

or,

$$\left| \int_a^x \int_a^b \eta(y)\, d_y\, p(x, y) - \sum_{j=1}^n \eta(y_j') \int_a^x [\, p(x, y_j) - p(x, y_{j-1})\,]\, dx \right| \leqq \epsilon/2.$$

It is true further that

$$\left| \int_a^b \eta(y)\, d_y \int_a^x p(x, y)\, dx \right.$$
$$\left. - \sum_{j=1}^n \eta(y_j') \int_a^x [\, p(x, y_j) - p(x, y_{j-1})\,]\, dx \right| \leqq \epsilon/2.$$

Hence,

$$\left| \int_a^x \int_a^b \eta(y)\, d_y\, p(x, y)\, dx - \int_a^b \eta(y)\, d_y \int_a^x p(x, y)\, dx \right| \leqq \epsilon,$$

and since these integrals are independent of ϵ, we have the desired equality.

THEOREM 10. *If $\eta(x)$ is of class D' on ab and p, q, r have the properties described for p, the integral*

$$(17)\quad I(\eta) = \iint_S \{\eta(x)\eta(y)\, d_{xy}p + 2\eta(x)\eta'(y)\, d_{xy}q + \eta'(x)\eta'(y)\, d_{xy}r\}$$

is well defined and a function $\eta_1(x)$ of class C' can be chosen so that

$$\eta_1(a) = \eta(a), \qquad \eta_1(b) = \eta(b), \qquad |I(\eta_1) - I(\eta)| < \epsilon,$$

where ϵ is an arbitrarily assigned positive quantity.

* H. E. Bray, loc. cit., p. 179.

Let the corners of the curve $\eta(x)$ be rounded off to form a function $\eta_1(x)$ in such a way that each interval in which η differs from η_1 has length less than 2δ. Then the portions of S in which the products $\eta(x)\eta(y)$ and $\eta_1(x)\eta_1(y)$ differ consist of strips of width 2δ parallel to the x-axis and to the y-axis. If the numbers of strips parallel to the two axes are r and s respectively, then the proof used in Theorem 7 shows that the integrals

$$\int\int_S \eta(x)\eta(y)\,d_{xy}\,p, \qquad \int\int_S \eta_1(x)\eta_1(y)\,d_{xy}\,p$$

differ by less than $4(r+s)M^2V\delta$. A similar argument applies to the second and third parts of (17), and since δ is arbitrary the theorem is established.

In the sequel we shall be interested in the solution u of a linear functional equation of the form

$$L(u; x) = kx + l, \tag{18}$$

where k, l are constants. The functional $L(u; x)$ is supposed to be single-valued when $u(\xi)$, x are given, and linear in the argument u. We wish to study some properties of this equation when it has a unique solution for each k, l.

THEOREM 11. *If $L(u; x) = kx + l$ has a unique solution $u(x)$ of class C' for each k, l, then the equation $L(u; x) = 0$ has only the solution $u \equiv 0$, and there exist two linearly independent solutions u_1, u_2 of the original equation such that the determinant*

$$\begin{vmatrix} L'(u_1; x) & L'(u_2; x) \\ L(u_1; x) & L(u_2; x) \end{vmatrix}$$

is different from zero.

By hypothesis the equation (18) has a unique solution for each k, l and therefore a unique solution for $k = 0$, $l = 0$. But from the properties of a linear functional it is clear that $u \equiv 0$ satisfies the equation $L(u; x) = 0$, and therefore this equation has only the solution $u \equiv 0$. Let u_1 be the unique solution of (18) for $k = 1$, $l = 0$; u_2 the solution for $k = 0$, $l = 1$. Then

$$\begin{vmatrix} L'(u_1; x) & L'(u_2; x) \\ L(u_1; x) & L(u_2; x) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ x & 1 \end{vmatrix} \neq 0.$$

Conversely, we can prove

THEOREM 12. *If $L(u; x) = 0$ has only the solution $u \equiv 0$ and if u_1, u_2 are two solutions of $L(u; x) = kx + l$ such that the determinant*

$$\begin{vmatrix} L'(u_1; x) & L'(u_2; x) \\ L(u_1; x) & L(u_2; x) \end{vmatrix}$$

is different from zero, then the equation $L(u; x) = kx + l$ has one and only one solution for each k, l.

Let k_1, l_1 be the constants corresponding to the solution u_1, and k_2, l_2 the constants corresponding to the solution u_2. Then

$$\begin{vmatrix} L'(u_1; x) & L'(u_2; x) \\ L(u_1; x) & L(u_2; x) \end{vmatrix} = \begin{vmatrix} k_1 & k_2 \\ k_1 x + l_1 & k_2 x + l_2 \end{vmatrix} = \begin{vmatrix} k_1 & k_2 \\ l_1 & l_2 \end{vmatrix} \neq 0 .$$

Hence the equations

$$c_1 k_1 + c_2 k_2 = k,$$

$$c_1 l_1 + c_2 l_2 = l$$

can be solved uniquely for c_1, c_2, and $u = c_1 u_1 + c_2 u_2$ satisfies the equation (18). Moreover it is the only solution because of the hypothesis that $L(u; x) = 0$ has only the solution $u \equiv 0$.

5. The second variation of $F(\lambda)$

The purpose of this section is to study the second variation of a functional $F(\lambda)$. To this end we shall be concerned with

(1) An arc λ_0 of class C' on the interval $x_1 \leqq x \leqq x_2$ joining the points $(x_1 y_1)$ and $(x_2 y_2)$.

(2) A functional $F(\lambda)$ defined on all arcs of class C' in a neighborhood $(\lambda_0)'_\delta$ of order one, and having first and second differentials at λ_0 as in § 3, with p, q', q'', r having properties described for p in § 4. The functions p and r may be taken symmetric without loss of generality.

At the end of § 3 it was proved that if $F(\lambda_0)$ is a minimum then the second differential $B(\eta, \eta)$ must be positive or zero for every function $\eta(x)$ of class C' on the interval $x_1 x_2$ such that $\eta(x_1) = \eta(x_2) = 0$.

We shall now consider the problem of minimizing the second variation. To this end we compute the first differential of $B(\eta, \eta)$. Written out in full,

$$(19)\qquad \begin{aligned} B(\eta, \eta) = \int_{x_1}^{x_2}\int_{x_1}^{x_2} [\, \eta(x)\eta(y) d_{xy}\, p(x, y) \\ + 2\eta(x)\eta'(y) d_{xy}\, q(x, y) + \eta'(x)\eta'(y) d_{xy}\, r(x, y)\,], \end{aligned}$$

where

$$q(x, y) = \tfrac{1}{2}[\, q'(y, x) + q''(x, y)\,].$$

If we give to η, η' the increments ζ, ζ' and compute the first differential of (19), we obtain

$$2\int_{x_1}^{x_2} [\, \zeta(x) d\omega(x) + \zeta'(x) d\omega_1(x)\,],$$

where

$$(20)\qquad \begin{aligned} \omega(x) &= \int_{x_1}^{x_2} [\, \eta(y) d_y\, p(x, y) + \eta'(y) d_y\, q(x, y)\,], \\ \omega_1(x) &= \int_{x_1}^{x_2} [\, \eta(y) d_y\, q(y, x) + \eta'(y) d_y\, r(x, y)\,]. \end{aligned}$$

From the results of § 2 it follows that *if a function* $\eta(x)$ *of class* D' *with* $\eta(x_1) = \eta(x_2) = 0$ *minimize the expression* (19) *for the second variation it must satisfy the equation*

$$L(\eta; x) \equiv \omega_1(x) - \int_{x_1}^{x} \omega(x)\,dx = kx + l, \qquad x_1 \leqq x \leqq x_2.$$

We make the following hypotheses on $L(\eta; x)$ which is defined for every $\eta(x)$ of class D' on $x_1 x_2$:

(1) It has a reciprocal $L_1(\eta; x)$ such that for every η of class C' on $x_1 x_2$

$$LL_1(\eta; x) = L_1 L(\eta; x) = \eta(x).$$

(2) When η is of class C' so is $L_1(\eta; x)$.

(3) There exists a constant A such that

$$|L(\eta; x)| \leqq AM, \qquad |L'(\eta; x)| \leqq AM,$$

where M is the maximum of $|\eta|$ and $|\eta'|$ on $x_1 x_2$.

Theorem 13. *If* $L(\eta; x)$ *has the properties described above and if* $F(\lambda_0)$ *is a minimum then no solution* $u(x)$ *of class* C' *of the equation*

$$L(u; x) = kx + l \tag{21}$$

can exist vanishing at x_1 *and a point* x_1' *between* x_1 *and* x_2 *but not identically zero between* x_1 *and* x_1', *and having* $u'(x_1') \neq 0$.

We shall first prove the

Lemma 5. *If* $u(x)$ *is a solution of equation* (21) *with the properties described in the theorem, then for the function*

$$\begin{aligned} \eta(x) &= u(x), \qquad & x_1 \leqq x \leqq x_1', \\ &\equiv 0, & x_1' \leqq x \leqq x_2, \end{aligned}$$

the value of $B(\eta, \eta)$ *is zero.*

By Theorem 8 of § 4 the second variation can be written in the form

$$\begin{aligned} B(\eta, \eta) &= \int_{x_1}^{x_2} \eta(x)\,d_x \int_{x_1}^{x_2} [\eta(y)\,d_y\,p(x, y) + \eta'(y)\,d_y\,q(x, y)] \\ &\qquad + \int_{x_1}^{x_2} \eta'(x)\,d_x \int_{x_1}^{x_2} [\eta(y)\,d_y\,q(y, x) + n'(y)\,d_y\,r(x, y)] \\ &= \int_{x_1}^{x_2} [\eta(x)\,d\omega(x) + \eta'(x)\,d\omega_1(x)]. \end{aligned}$$

Substitute for $\eta(x)$ the function defined in the theorem. Then

$$\begin{aligned} B(\eta, \eta) &= \int_{x_1}^{x_1'} [u(x)\,d\omega(x) + u'(x)\,d\omega_1(x)] \\ &= [u(x)\,\omega(x)]_{x_1}^{x_1'} + \int_{x_1}^{x_1'} u'(x)\,d\left[\omega_1(x) - \int_{x_1}^{x} \omega(x)\,dx\right]. \end{aligned}$$

The terms outside the integral sign vanish since $u(x_1) = u(x_1') = 0$. By hypothesis $u(x)$ satisfies the equation (21), and therefore

$$\int_{x_1}^{x_1'} u'(x)\, d\left[\omega_1(x) - \int_{x_1}^{x} \omega(x)\, dx\right] = k \int_{x_1}^{x_1'} u'(x)\, dx = k \cdot u(x)\Big]_{x_1}^{x_1'} = 0,$$

which proves the lemma.

Hence the function $\eta(x)$ thus defined gives the second variation the value zero, and this is the smallest possible value for $B(\eta, \eta)$ if $F(\lambda)$ is a minimum at λ_0, as has been proved. We can show, however, that in case there exists such a function $\eta(x)$ making $B(\eta, \eta)$ vanish, then there will surely be others which make it negative. For this purpose write

$$\begin{aligned} B(\eta, \eta) = \int\int_S \{\eta(x)\eta(y)\, d_{xy}\, p^* + 2\eta(x)\eta'(y)\, d_{xy}\, q \\ + \eta'(x)\eta'(y)\, d_{xy}\, r\} - h\int_{x_1}^{x_2} \eta^2(x)\, dx \\ \equiv B^*(\eta, \eta) - h\int_{x_1}^{x_2} \eta^2(x)\, dx, \end{aligned}$$

with p^* defined by the conditions,

$$\begin{aligned} p^*(x, y) &= p(x, y) + h(y - x) \qquad \text{for} \quad x_1 \leqq y \leqq x, \\ &= p(x, y) \qquad \text{for} \quad x \leqq y \leqq x_2. \end{aligned}$$

If we denote by $L^*(\eta; x)$ the result obtained by replacing in $L(\eta; x)$ the function $p(x, y)$ by $p^*(x, y)$ we have the relation

$$L^*(\eta; x) = L(\eta; x) - h\int_{x_1}^{x}\int_{x_1}^{x} \eta(y)\, dy dx.$$

LEMMA 6. *For sufficiently small values of h the equation*

$$L^*(u; x) = kx + l \tag{22}$$

has a solution $u_1(x, h)$ of class C' on $x_1 x_2$ corresponding to $k = 1$, $l = 0$, and a solution $u_2(x, h)$ of class C' on $x_1 x_2$ corresponding to $k = 0$, $l = 1$. These solutions are continuous in a domain $x_1 \leqq x \leqq x_2$, $|h| \leqq \delta$, and are linearly independent.

According to hypothesis (1) the equation

$$L(u; x) = v(x), \tag{23}$$

where u, v are of class C' on $x_1 x_2$ has a unique solution $u(x)$. If we apply the operation L_1 to the members of the equation

$$L^*(u; x) = v(x), \tag{24}$$

we obtain

$$u(x,h) - hL_1\left(\int_{x_1}^{x}\int_{x_1}^{x} u(y)\,dydx; x\right) = L_1(v; x) = \phi(x),$$

or,

$$(25) \qquad u(x,h) = \phi(x) + hL_2(u; x),$$

where L_2 has the properties (2) and (3) of L_1 and $\phi(x)$ is of class C'. Repeated applications of the last equation give the series

$$(26) \quad u(x,h) = \phi(x) + hL_2(\phi; x) + h^2 L_2^2(\phi; x) + \cdots + h^n L_2^n(\phi; x) + \cdots.$$

By property (2) each term of this series is of class C'. By (3) the terms of the series formed of the x-derivatives of (26) as well as the terms of the series itself are dominated by the terms of the series,

$$M(1 + hA(x_2 - x_1)^2 + h^2 A^2 (x_2 - x_1)^4 + \cdots + h^n A^n (x_2 - x_1)^{2n} + \cdots),$$

whence it is seen that the two series are uniformly convergent for $x_1 \leqq x \leqq x_2$, $|h| \leqq \delta$, when δ is sufficiently small.

The series (26) satisfies equation (25) and therefore (24) as we see by operating on (26) with L_2. To justify this last statement write (26) in the form $u = s_n + r_n$, where s_n is the sum of the first n terms. Then

$$|L_2(u) - L_2(s_n)| = |L_2(r_n)| \leqq \epsilon'.$$

Also (26) is the only solution. For if there were two solutions, their difference $\psi(x)$ would satisfy the equation

$$(27) \qquad \psi(x) = hL_2(\psi; x).$$

This equation can have no solution other than $\psi \equiv 0$ for $h \leqq \delta$ when δ is sufficiently small. For the functions $h^n L_2^n(\phi; x)$ in (26) tend to zero if $|h| \leqq \delta$; but if $\psi(x,h)$ satisfies (27) all these terms are identical with $\phi(x) \equiv 0$, and therefore $\psi \equiv 0$.

These results together with Theorem 11 of § 4 prove the lemma.

LEMMA 7. *If equation* (21) *has a solution* $u(x)$ *as described in the theorem then for sufficiently small values of* $|h|$ *the equation* (22) *has a solution* $u(x,h)$ *with similar properties.*

Let $u(x)$ be the solution of (21) with constants k, l. Then by Theorems 11 and 12 of § 4 it is expressible in the form

$$u(x) = ku_1(x,0) + lu_2(x,0).$$

Since $u(x)$ vanishes at x_1 it may be that both $u_1(x_1,0)$ and $u_2(x_1,0)$ are zero; in which case

$$u(x,h) = ku_1(x,h) + lu_2(x,h)$$

is the desired solution. If $u_1(x_1,0)$ and $u_2(x_1,0)$ are not both zero, then

$$u(x_1) = ku_1(x_1, 0) + lu_2(x_1, 0) = 0,$$

or,

$$k : l = -u_2(x_1, 0) : u_1(x_1, 0);$$

and

$$u(x, h) = \begin{vmatrix} u_1(x, h) & u_2(x, h) \\ u_1(x_1, h) & u_2(x_1, h) \end{vmatrix}$$

is the solution demanded. For, from the hypothesis that $u'(x_1')$ is different from zero it follows that $u_1(x_1' - \delta)$ and $u_1(x_1' + \delta)$ will have opposite signs for a sufficiently small δ. The same will be true of $u(x_1' - \delta, h)$ and $u(x_1' + \delta, h)$ for sufficiently small $|h|$. Hence $u(x, h)$ for such values of h will surely vanish at least once between $x_1' - \delta$ and $x_1' + \delta$. The value x_h below can be selected as the first zero of $u(x, h)$ after $x_1' - \delta$.

To prove Theorem 13, choose $h > 0$, and

$$\begin{aligned} \eta(x) &= u(x, h) \qquad \text{for} \qquad x_1 \leqq x \leqq x_h, \\ &\equiv 0 \qquad\qquad\ \ \text{for} \qquad x_h \leqq x \leqq x_2, \end{aligned}$$

where x_h is a zero of $u(x, h)$ between x_1 and x_2. Then

$$B(\eta, \eta) = B^*(\eta, \eta) - h\int_{x_1}^{x_2} \eta^2(x)\,dx = -h\int_{x_1}^{x_2} \eta^2(x)\,dx < 0,$$

since $B^*(\eta, \eta) = 0$ for the η just chosen. Finally, if $B(\eta, \eta) < 0$ for an η of class D' it will also be less than zero for an arc of class C' as is seen by applying Theorem 10 of § 4.

6. Application to the calculus of variations

The purpose of this part of the paper is to interpret the foregoing results in the case of the functionals of the calculus of variations.

For the simplest problem of the calculus of variations the functional $F(\lambda)$ has the form

$$F(\lambda) = \int_{x_1}^{x_2} f(x, \lambda, \lambda')\,dx. \tag{28}$$

We have seen that the functions $u(x)$ and $u_1(x)$ occurring in the expression (3) of the first variation of $F(\lambda)$ are not uniquely determined by the $F(\lambda)$, since the integrals are unaltered if $u(x)$ and $u_1(x)$ are each increased or diminished by a constant; hence we are at liberty to assign to them an arbitrary value, say zero, at a particular point of the interval $x_1 x_2$. We may then make $u(x)$ and $u_1(x)$ vanish at x_1. If for the functional $F(\lambda)$ in (28) we define $u(x)$ and $u_1(x)$ as follows:

$$u(x) = \int_{x_1}^{x} f_\lambda\,dx, \qquad u_1(x) = \int_{x_1}^{x} f_{\lambda'}\,dx, \tag{29}$$

then the expression (3) for its first variation assumes the familiar form

$$\int_{x_1}^{x} (f_\lambda \eta + f_{\lambda'} \eta')\, dx .$$

We have derived as a first necessary condition for a minimum of $F(\lambda)$ the relation

$$\hat{u}'(x) = k,$$

where

$$\hat{u}(x) = u_1(x) - \int_{x_1}^{x} u(x)\, dx .$$

For the functions in (29) this condition is equivalent to the relation

$$f_{\lambda'} - \int_{x_1}^{x} f_\lambda\, dx = k,$$

which upon differentiation leads to the Euler equation

$$f_\lambda - \frac{d}{dx} f_{\lambda'} = 0 .$$

Since the function $u(x)$ defined in (29) is continuous the conditions (10) on $u(x)$ and $u_1(x)$, which were found to hold if the first variation vanishes for every $\eta(x)$ of class C' whether or not $\eta(x_1) = \eta(x_2) = 0$, become the transversality conditions

$$f_{\lambda'}(x_1) = f_{\lambda'}(x_2) = 0 .$$

Let us next consider the second variation $B(\eta, \eta)$ of $F(\lambda)$ when $F(\lambda)$ is given by (28). Again, the functions p, q, r, occurring in $B(\eta, \eta)$ are not uniquely determined by $F(\lambda)$, for the value of the double integral (19) is unaltered by the addition of an arbitrary function of x alone or an arbitrary function of y alone. The functions may therefore be chosen identically zero for a particular value of x and a particular value of y. Define the function $p(x, y)$ by the conditions

$$\begin{aligned} p(x, y) &= \int_{x_1}^{y} P\,dy \qquad \text{for} \qquad x_1 \leqq y \leqq x, \\ &= \int_{x_1}^{x} P\,dy \qquad \text{for} \qquad x \leqq y \leqq x_2, \end{aligned} \tag{30}$$

where $P = f_{\lambda\lambda}$, and make similar definitions for $q(x, y)$ and $r(x, y)$ in terms of $Q = f_{\lambda\lambda'}$ and $R = f_{\lambda'\lambda'}$, respectively. The functions p, q, r so defined have all the properties assumed for p in § 4.

If the functions defined above are substituted in the expression for $B(\eta, \eta)$ the double integral (19) is reduced to the single integral

$$\int_{x_1}^{x_2} (\eta^2 P + 2\eta\eta' Q + \eta'^2 R)\,dx.$$

The first member of the equation

$$L(\eta; x) = \omega_1(x) - \int_{x_1}^{x} \omega(x)\,dx = kx + l, \tag{21}$$

where ω, ω_1 are defined by the relations (20), assumes for the p, q, r in (30) the form

$$\begin{aligned}\omega_1(x) - \int_{x_1}^{x} \omega(x)\,dx &= \int_{x_1}^{x} \{\eta(y)[Q(y) - (x-y)P(y)]\,dy \\ &\qquad + \eta'(y)[R(y) - (x-y)Q(y)]\,dy\} \\ &= \int_{x_1}^{x} \eta(y)\{(x-y)[Q'(y) - P(y)] \\ &\qquad - R'(y)\}\,dy + \eta(x)R(x).\end{aligned}$$

Accordingly, the equation (21) reduces to a Volterra integral equation of the first kind and has therefore a unique solution for every $v(x)$ of class C', provided $R(x) \neq 0$. The properties assumed for $L(\eta; x)$ in § 5 are here justifiable. Equation (21) differentiated twice gives the Jacobi differential equation,

$$\eta(x)[Q'(x) - P(x)] + \frac{d}{dx}[\eta'(x)R(x)] = 0.$$

7. The character of the operation $L(u; x)$

The analog of the Jacobi condition of the calculus of variations which has been obtained as a second necessary condition for a minimum of $F(\lambda)$ is expressed in terms of a solution of the equation

$$\begin{aligned}&\int_{x_1}^{x_2} [u(y)\,d_y q(y, x) + u'(y)\,d_y r(x, y)] \\ &\qquad - \int_{x_1}^{x}\int_{x_1}^{x_2} [u(y)\,d_y p(x, y) + u'(y)\,d_y q(x, y)]\,dx = kx + l.\end{aligned} \tag{21'}$$

The first member of this equation is a linear functional $L(u; x)$ and with the help of the theorems of § 4 is expressible as a Stieltjes integral. Theorem 6 and the conditions $u(x_1) = u(x_2) = 0$ enable us to replace the two integrals

$$\int_{x_1}^{x_2} u'(y)\,d_y r(x, y), \qquad \int_{x_1}^{x_2} u'(y)\,d_y q(x, y)$$

by the integrals

$$-\int_{x_1}^{x_2} u(y)\,d_y r_2(x, y), \qquad -\int_{x_1}^{x_2} u(y)\,d_y q_2(x, y),$$

where $r_2(x, y)$ and $q_2(x, y)$ have the properties defined in § 4. Now apply Theorem 9 and we see that equation (21′) reduces to the form

$$\int_{x_1}^{x_2} u(y)\, d_y K(x, y) = kx + l,$$

where

$$K(x, y = q(y, x) - r_2(x, y) - \int_{x_1}^{x} [p(x, y) - q_2(x, y)]\, dx.$$

If the only discontinuity of the function $r_2(x, y)$ occurs on the diagonal of the square S and is there equal to $R(x)$, the equation will have the form

$$\int_{x_1}^{x_2} u(y)\, d_y K_1(x, y) - R(x)u(x) = kx + l, \tag{31}$$

where $K_1(x, y)$ is continuous and of limited variation in each variable uniformly with respect to the other.

We can show that the integral in (31) represents a transformation which has the property of complete continuity.* A transformation is said to be completely continuous if it transforms a bounded sequence into a "compact" sequence, that is, into a sequence such that every subsequence of itself contains a further subsequence which is uniformly convergent. A necessary and sufficient condition that a bounded sequence $\{\phi_n\}$ be compact is that for a positive ϵ there exists a δ such that for $|x - x'| < \delta$ and for all ϕ_n the inequality

$$|\phi_n(x) - \phi_n(x')| < \epsilon$$

holds.†

Let $\{u_n\}$ be a bounded sequence of continuous functions such that $|u_n| < G$. The integral

$$\phi(x) = \int_{x_1}^{x_2} u(y)\, d_y K_1(x, y)$$

is a continuous function of x.‡ The total variation with respect to y of the function $\{K_1(x, y) - K_1(x', y)\}$ is the upper bound in the set of continuous functions $u(y)$ of the expression§

$$\frac{\left| \int_{x_1}^{x_2} u(y)\, d_y [K_1(x, y) - K_1(x', y)] \right|}{M(u)}.$$

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* F. Riesz, *Ueber lineare Funktionalgleichungen*, Acta Mathematica, vol. 41: 1 (1916), p. 73.

† C. Arzelà, *Sulle funzioni di linee*, Memorie d. R. Accad. d. Scienze di Bologna, ser. 5, vol. V (1895), S. 225–244. F. Riesz, loc. cit., p. 93.

‡ H. E. Bray, loc. cit., p. 180.

§ F. Riesz, *Sur certains systèmes singuliers d'équations intégrales*, Annales scientifiques de l'école normale supérieure, vol. 28 (1911), p. 43; Fréchet II p. 217.

But on account of the continuity of $\phi(x)$ this last expression may be made less than ϵ/G by taking $|x - x'| < \delta$. Hence the total variation with respect to y of $\{K_1(x, y) - K_1(x', y)\}$ is less than ϵ/G for $|x - x'| < \delta$. We have then the relation

$$|\phi_n(x) - \phi_n(x')| = \left| \int_{x_1}^{x_2} u_n(y) d_y [K_1(x, y) - K_1(x', y)] \right| < \epsilon .$$

F. Riesz discusses in his article on linear functional equations referred to above the inversion of a transformation of the form $E - A$, where E is the identical transformation and A is a completely continuous transformation. The results of his article are applicable to the equation (31) if $R(x) \neq 0$.

THE UNIVERSITY OF KENTUCKY

INVARIANTS OF INFINITE GROUPS IN THE PLANE*

BY

EUGENE FRANCIS SIMONDS

In a previous paper published in these Transactions† the writer discussed the general question of the invariants of differential configurations in the plane under groups—finite and infinite—of point or contact transformations. The present paper deals with the application of the general results therein obtained to the various types of infinite groups of point and contact transformations.

I. Point transformations

The problem of the determination of the different types of such groups was solved by Lie.‡ Adopting his results as a basis of classification we subdivide infinite groups of point transformations into the following types:

(A) The entire group of point transformations.

(B) Those reducible by change of coördinates to the group which multiplies areas by a constant.

(C) Those reducible to the area-preserving group.

(D) Those which leave invariant one—and only one—differential equation of the first order.

(E) Those which leave invariant two differential equations of the first order.

The types (A) and (C) have already been dealt with from the point of view of their invariants. It was found that for type (A) the smallest number of curves—denoted by λ_n—having contact of zero order necessary for the existence of an invariant of order n was $2n + 2$, while for (C) it was $n + 3$. Explicit expressions for such invariants were calculated as far as the third order.

We now proceed to consider the remaining types.

Type (B). This is the only class of infinite groups of point transformations defined exclusively by differential equations of the second order. Its simplest representative is the group

$$\xi_{xx} + \eta_{xy} = 0; \qquad \xi_{xy} + \eta_{yy} = 0,$$

which transforms areas in a constant ratio.

* Presented to the Society, October 25, 1919.

† Vol. 19 (1918), pp. 223–250.

‡ *Ueber unendliche continuirliche Gruppen*, Videnskabs-Selskabet Christiania, 1883.

The equivalent complete system of linear differential equations does not differ from that of the group (A) until it is extended to the third order. The additional equations corresponding to the third and higher orders are identical with those of the area-preserving group. *Hence for the type* (B) *we have* $\lambda_1 = 4$; $\lambda_2 = 6$; $\lambda_n = n + 3\,(n > 2)$.

It is a simple matter to obtain the first three invariants. The first two are identical with those of the group (A).* If we compare the complete system with that of the area-preserving group we find that the only difference is that the equation

$$V \equiv \sum 2y_i' \frac{\partial f}{\partial y_i'} + \sum 3y_i'' \frac{\partial f}{\partial y_i''} + \sum 4y_i''' \frac{\partial f}{\partial y_i'''} = 0,$$

characteristic of the area-preserving group is replaced by the pair of equations

$$U_1 \equiv \sum y_i' \frac{\partial f}{\partial y_i'} + \sum y_i'' \frac{\partial f}{\partial y_i''} + \sum y_i''' \frac{\partial f}{\partial y_i'''} = 0,$$

$$U_2 \equiv \sum y_i' \frac{\partial f}{\partial y_i'} + \sum 2y_i'' \frac{\partial f}{\partial y_i''} + \sum 3y_i''' \frac{\partial f}{\partial y_i'''} = 0.$$

Since $V = U_1 + U_2$, it is clear that the group (B) differs only from the area-preserving group by the addition of the equation $U_1 = 0$. We have thus proved the

THEOREM. *The invariants of the group* (B) *are those invariants of the area-preserving group which are homogeneous and of degree zero.*

Of the three invariants of the area-preserving group I_4, J_5, K_6—given in our previous paper—I_4 was homogeneous of degree zero; J_5 was homogeneous of degree $-\frac{1}{3}$; and K_6 was homogeneous of degree -1. It is clear, then, that

$$K_6 J_5^{-3}$$

is a third-order invariant of the group (B).

It is to be remarked that in the expression

$$J_5^{-6}(A_6 K_6 + A_7 K_7 + A_8 K_8)$$

for the third-order invariant of the entire group (A)† each of the three terms

$$J_5^{-6} A_6 K_6, \qquad J_5^{-6} A_7 K_7, \qquad J_5^{-6} A_8 K_8$$

is a third-order invariant of the group (B).

Type (D). The canonical form of the equation-system defining this type is the single equation $\xi_y = 0$, and the finite equations of the group are

$$X = \phi(x), \qquad Y = \psi(x, y),$$

ϕ and ψ being arbitrary functions.

* Page 230.

† Page 244.

The most important subgroup not of type (E) is

$$X = \phi(x), \qquad Y = \frac{\psi_1(x) + \psi_2(x)y}{\psi_3(x) + \psi_4(x)y}.$$

This was shown by the writer to be the largest group having the following properties:

1. For the existence of an infinite system of absolute invariants only a finite number of curves are required.* (In this case $\lambda_n = 5$, which is the highest value of λ_n independent of n for any group.)

2. Though the base-points of the configuration may have only one coördinate in common, invariants exist under the group.†

For type (D) we shall content ourselves with merely stating the following results, which are readily verified.

THEOREM. *For the most general group leaving invariant a differential equation of the first order* $\lambda_1 = 3$. *For* $n > 1$, $\lambda_n = n + 3$.

If the invariant differential equation is

$$y' = f(x, y)$$

the invariant of the first order is obviously the cross-ratio $[y_1' \, y_2' \, y_3' f]$.

We have already found that in case all the arbitrary functions in the defining-equations of a group are functions of one variable, the value of λ_n for the group could at once be written down. There are then only three cases left under type (D), corresponding respectively to the values

$$x, \qquad ax + b, \qquad \frac{ax + b}{cx + d}$$

of $\phi(x)$, the second function $\psi(x, y)$ being completely arbitrary.

For all of these cases λ_n *takes the value* $n + 2$ *after a finite value of* n. This is an illustration of the result which is true in general that the presence of a finite number of arbitrary constants in the defining-equations of the group can only affect λ_n for a finite number of values of n. The subsequent values of λ_n depend only on the number of arbitrary *functions*.

Type (E). This last type includes some important groups, the most noteworthy being the conformal. The equilong group may be regarded as a degenerate case. Certain results have been obtained for the conformal and equilong groups by Kasner,‡ and for the equilong by the writer.§

* Page 235.

† Page 236.

‡ *Conformal Geometry, Proceedings of the Fifth International Congress*, Cambridge (1912) vol. 2, pp. 81–87; *Conformal classification of analytic arcs or elements: Poincaré's local problem of conformal geometry*, these Transactions, vol. 16 (1915), pp. 333–349; *Equilong invariants and convergence proofs*, Bulletin of the American Mathematica Society, vol. 23 (1917), pp. 341–347.

§ Loc. cit., pp. 244–246.

The general type of these groups is

$$X = \phi(x), \qquad Y = \psi(y), \tag{1}$$

ϕ and ψ being arbitrary functions. The conformal group can be reduced to the form (I) by the transformation

$$x = u + iv, \qquad y = u - iv.$$

It is found that for the largest groups of type (E)*—that is, those involving two arbitrary functions—*$\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_n = 3$ *for* $n > 2$. This is true of both the conformal and equilong groups. In the latter case the three curves of the configuration need not pass through the same point, though for the most general type it is necessary that they do so.* We shall obtain invariants of the group (I) as far as the fourth order.†

The equations (1) may be taken in the form

$$X = \int e^{-\phi(x)}\,dx, \qquad Y = \int e^{\psi(y)}\,dy.$$

We obtain in succession

$$Y' = e^{\psi+\phi}\,y',$$

$$Y'' = e^{\psi+2\phi}(y'' + \psi' y'^2 + \phi' y'),$$

$$Y' Y''' - \tfrac{3}{2}Y''^2 = e^{2\psi+4\phi}[(y' y''' - \tfrac{3}{2}y''^2) + (\psi'' - \tfrac{1}{2}\psi'^2)y'^4 + (\phi'' + \tfrac{1}{2}\phi'^2)y'^2],$$

$$\begin{aligned} Y'^2 Y^{(\mathrm{iv})} - 6Y' Y'' Y''' + 6Y''^3 = e^{3\psi+5\phi}[&(y'^2 y^{(\mathrm{iv})} - 6y' y'' y''' + 6y''^3) \\ &- 2\psi' y'^2(y' y''' - \tfrac{3}{2}y''^2) - 2y'^2(y'' + \psi' y'^2)(\phi'' + \tfrac{1}{2}\phi'^2) \\ &+ (\psi''' + \psi'^3 + 3\psi'\psi'')y'^6 + (\phi''' + \phi'\phi'')y'^3].\end{aligned}$$

The presence of the exponential factors on the right means that all the invariants are homogeneous and isobaric. This is also evident from the consideration that the groups $X = x$, $Y = ay$; $X = ax$, $Y = y$ are subgroups of (I). The first has only homogeneous invariants; the second only isobaric.

By eliminating the arbitrary functions from the above equations we find a relative invariant of each order for three curves passing through a common

* Loc. cit., p. 235.

† The fact that four curves through a common point, with distinct tangents, have a conformal invariant of second order was first stated in an article by Kasner, *The geometry of differential elements of the second order, etc.*, American Journal of Mathematics, vol. 28 (1906), p. 213. The invariant is there shown to be expressible as the anharmonic ratio of the lines connecting C_1 with C_2, C_3, C_4, O, where O is the common point, and C_1, C_2, C_3, C_4 are the centers of curvature of the four curves.

point. These are

$$I_1 = y_i', \qquad I_2 = |y_i'' \, y_i'^2 \, y_i'|,$$

$$I_3 = |y_i' \, y_i''' - \tfrac{3}{2} y_i''^2 \, y_i'^4 \, y_i'^2|,$$

$$I_4 = |y_i'^2 \, y_i^{(\mathrm{iv})} - 6 y_i' \, y_i'' \, y_i''' + 6 y_i''^3 - 2K_1 \, y_i'^2 (y_i' \, y_i''' - \tfrac{3}{2} y_i''^2)$$
$$- 2K_2 (y_i'' + K_1 \, y_i'^2) \, y_i'^6 \, y_i'^3|.$$

I_2, I_3, I_4 are three-rowed determinants, of which the ith row is given in each case $(i = 1, 2, 3)$. Moreover

$$K_1 = \begin{vmatrix} y_1' \, y_1'' \\ y_2' \, y_2'' \end{vmatrix} \div y' \, y' (y' - y'),$$

$$K_2 = \begin{vmatrix} y_1'^4 \, y_1' \, y_1''' - \tfrac{3}{2} y_1''^2 \\ y_2'^4 \, y_2' \, y_2''' - \tfrac{3}{2} y_2''^2 \end{vmatrix} \div y_1'^2 \, y_2'^2 (y_1'^2 - y_2'^2).$$

The magnifying factors of I_1, I_2, I_3, I_4 are respectively $e^{\psi+\phi}$, $e^{4\psi+5\phi}$, $e^{8\psi+10\phi}$, $e^{12\psi+14\phi}$. Hence we have the four absolute invariants

$$y_2' - y_1'; \qquad y_3' - y_1'; \qquad I_3 I_2^{-2}; \qquad I_4 I_3^{-1} I_1^{-4}.$$

II. Contact transformations

In our previous paper we dealt with the question of finite groups. There remain only those infinite groups that are not reducible to groups of point transformations. There are three types, whose characteristic functions are

$$(F) \quad ay + W(x, y'),$$

$$(G) \quad W(x, y'),$$

$$(H) \quad \Omega(x, y, y'),$$

where W and Ω are arbitrary functions of their arguments, and a is an arbitrary parameter.* (F) and (G) are imprimitive. (H) is the entire group of contact transformations in the plane.

We have shown that the group defined by (H) has invariants of the same form as those of the entire group of point transformations, except that the order of each derivative is increased by unity. $2n$ analytic curves having contact of the first order but subject only to that restriction have a single invariant of order $n + 1$ for every $n > 1$.†

The infinitesimal transformations of the type (G) are

$$(2) \qquad \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \pi \frac{\partial f}{\partial y'}$$

* Lie, Leipsiger Abhandlungen, vol. 21, p. 150.
† Loc. cit., Transa, p. 247.

where

$$\xi = W_{y'}, \qquad \eta = y' W_{y'} - W, \qquad \pi = - W_x.$$

It is to be noticed that $\xi_x + \pi_{y'} = 0$. If, then, we take x, y' as point coördinates in a plane, the group

$$\xi \frac{\partial f}{\partial x} + \pi \frac{\partial f}{\partial y'}$$

becomes the area-preserving group of point transformations. The extended complete system of (2) is identical with that of (3), since neither ξ nor π involve y. It follows, then, that the complete system equivalent to the group of contact transformations whose characteristic function is the same as that of the area-preserving group of point transformations except that the order of each derivative is increased by unity.

For the entire group of contact transformations it is necessary that all the curves of the configuration shall pass through a common point and have contact of at least the first order if the configuration is to have invariants. For the group $W(x, y')$ this is not necessary. Since all the arbitrary functions involve only x and y', the most general configuration having invariants consists of curves whose base-points have the same abscissa, and whose tangents are parallel. From our results for the area-preserving group, then, we have the

THEOREM. *If our configuration consists of curves whose base-points have the same abscissa and whose tangents are parallel, $n + 3$ curves have an invariant of order $n + 1$ under the group of contact transformations whose characteristic function is $W(x, y')$. The invariants are of the same form as those of the area-preserving group of point-transformations for configurations of $n + 3$ curves having contact of zero order. The order of each derivative, however, is increased by unity.*

The group $ay + W(x, y')$ has the infinitesimal transformations

$$\xi = W_{y'}; \qquad \eta = y' W_{y'} - ay - W; \qquad \pi = - W_x - ay'.$$

Hence we have

$$\xi_{xx} + \pi_{xy'} = 0,$$

$$\xi_{xy'} + \pi_{y'y'} = 0,$$

and it is clear that this group bears the same relation to the group (B) of point transformations as $W(x, y')$ bears to the area-preserving group. The results for this group, then, are as follows:

THEOREM. *The group of contact transformations whose characteristic function is $ay + W(x, y')$ has invariants for a configuration of $n + 3$ curves whose base-points have the same abscissa and whose tangents are parallel. There is one invariant of order $n + 1$, which is identical with the corresponding invariant*

of the group of point transformations which multiplies areas by a constant, except that the order of each derivative is increased by unity.

And finally:

THEOREM. *The invariants of the group* $ay + W(x, y')$ *are the invariants of the group* $W(x, y')$ *for the same configuration which are homogeneous of degree zero.*

III. SUMMARY

We may resume the results for λ_n for the various types of infinite groups in the following table:

	Point Transformations					Contact		
Type	A	B	C	D	E	F	G	H
λ_n	$2n+2$	$n+3$	$n+3$	$n+2$	3	$2n+2$	$n+3$	$n+3$

UNIVERSITY OF SYDNEY, AUSTRALIA
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ON TRIPLY ORTHOGONAL CONGRUENCES*

BY

JAMES BYRNIE SHAW

Introduction

1. A vector field is defined by the value of a vector σ at each point of the field. This vector σ is a function of ρ, the vector to the point from some given origin. If σ is taken as tangent to a curve at each of the points of the space considered, these tangents will envelop a congruence of curves, the vector lines of the field of σ. The tensor or length of σ is not determinable from the congruence, the congruence depending only upon the unit vector $U\sigma$. The congruence itself however determines many properties of the field, which are of course purely geometric properties, and in physical phenomena are not dependent on physical facts except through the law by which $U\sigma$ is determined. It is purposed to consider some of these properties which belong to the congruences determined by three unit vectors which are everywhere mutually perpendicular. These will generally be indicated by α, β, γ; but in case they are the tangent, principal normal, and binormal of a curve, we will use ν with a proper subscript to indicate the principal normal, and μ with the same subscript to indicate the binormal. For instance, if we are discussing a curve of the α congruence the normal will be ν_a and the binormal μ_a.

The algebra used throughout is Quaternions, all the usual symbols of that algebra being introduced with no explanation. However it should be noted that such an expression as $\alpha\beta$ is always a quaternion product.

The symbol ϕ and the symbol θ are throughout linear vector operators,— in the case of ϕ related to certain vectors which are subscripts of ϕ. For instance we define for any vector η: $\phi_\eta = - S\,(\;)\,\nabla \cdot \eta$. The occurrence of such operators is due to the fact that if the point under discussion at the end of ρ is displaced to $\rho + d\rho$, the vector η becomes $\eta + d\eta$ where

$$d\eta = - Sd\rho\nabla \cdot \eta = \phi_\eta\, d\rho\,.$$

In every instance ∇ operates upon every variable that follows it unless it carries a subscript, in which case it operates upon everything with that subscript whether preceding or following, the subscripts being dropped after the differentiation.

* Presented to the Society, December 28, 1918.

2. We recall the Serret-Frenet formulas in this notation, indicating the curvature by c with proper subscript and the torsion by t with subscript.

For instance, we have for a curve of the α congruence

$$\nu_a c_a = \phi_a \alpha, \qquad \mu_a t_a - \alpha c_a = \phi_\nu \alpha, \qquad -\nu_a t_a = \phi_\mu \alpha,$$

since $\phi_a \alpha$ is the vector rate of change of α, $d\nu/ds$, in the direction of α, $\phi_\nu \alpha$ the rate of change, $d\nu/ds$, of ν, for displacements in the direction of α, etc. If we set $\omega_a = c_a \mu_a + t_a \alpha$ then ω_a is the rectifying line of the curve and we have (omitting the subscripts for brevity)

$$\frac{d\nu}{ds} = V\omega\nu, \qquad \frac{d\mu}{ds} = V\omega\mu, \qquad \frac{d\alpha}{ds} = V\omega\alpha.$$

In case we have $\eta = l\alpha + m\beta + n\gamma$ where l, m, n are constant and ω retains its meaning,

$$\frac{d\eta}{ds} = V\omega\eta.$$

THE SYSTEM α, β, γ

3. Since α, β, γ are unit vectors $S\alpha d\alpha = 0 = S\beta d\beta = S\gamma d\gamma$, or $S\alpha\phi_a d\rho = 0$ for all $d\rho$, and similarly for the others. Hence we have for any vector, $S\alpha\phi_a\,(\,) = 0$, or $\phi'_a \alpha = 0$ where ϕ' is the conjugate, or transverse operator, and likewise $\phi'_\beta \beta = 0$, $\phi'_\gamma \gamma = 0$.

Since $S\alpha\beta = 0$ etc. we have $S\alpha d\beta + S\beta d\alpha = 0$, or $S\alpha\phi_\beta\,(\,) + S\beta\phi_a\,(\,) = 0$, or $\phi'_\beta \alpha = -\phi'_a \beta$ with similar equations for the others.

Since $\alpha\beta = \gamma$ we have $\nabla\gamma = \nabla_1 \alpha_1 \beta + \nabla_1 \alpha\beta_1$, and since for any two vectors λ, μ we have always $\lambda\mu = -\mu\lambda + 2S\lambda\mu$ we obtain from the equation just written

$$\nabla\gamma = \nabla_1 \alpha_1 \beta - \nabla_1 \beta_1 \alpha + 2\nabla_1 S\alpha\beta_1,$$

or, remembering $\nabla_1 S\alpha\beta_1 = -\nabla_1 S\alpha_1 \beta$,

$$\nabla\gamma = \nabla_1 \alpha_1 \beta - \nabla_1 \beta_1 \alpha + 2\phi'_a \beta.$$

We obtain the similar equations for the other vectors α, β by interchanging the symbols cyclically. This equation shows that the quaternion $\nabla\gamma$ is fully determined from α and β. Taking the scalar and writing the expressions so that the subscripts may be omitted, the *convergence* of γ is

$$S\nabla\gamma = S\beta\nabla\alpha - S\alpha\nabla\beta, \qquad \text{or} \qquad S\gamma\,[\,V\alpha V\nabla\alpha + V\beta V\nabla\beta + V\gamma V\nabla\gamma\,].$$

The expression in the bracket being the same for all three vectors, we will set

$$2\epsilon = V\alpha V\nabla\alpha + V\beta V\nabla\beta + V\gamma V\nabla\gamma$$

and have the three convergences in the form

$$S\nabla\alpha = 2S\alpha\epsilon, \qquad S\nabla\beta = 2S\beta\epsilon, \qquad 2S\nabla\gamma = 2S\gamma\epsilon.$$

Hence

$$2\epsilon = -\alpha S\nabla\alpha - \beta S\nabla\beta - \gamma S\nabla\gamma.$$

Since

$$V\alpha V\nabla\alpha = S\alpha\nabla\cdot\alpha - \nabla_1 S\alpha_1\alpha = S\alpha\nabla\cdot\alpha - \tfrac{1}{2}\nabla S\alpha\alpha = -\phi_\alpha\alpha = -c_\alpha\nu_\alpha$$

it becomes evident that ϵ is half the sum of the three vector curvatures, with sign reversed, of the three curves for α, β, γ at the point considered.

4. We have, by taking the vector of $\nabla\gamma$, the *curl* of γ:

$$V\nabla\gamma = V\nabla_1\alpha_1\beta - V\nabla_1\beta_1\alpha + 2\phi_\alpha'\beta = \phi_\alpha\beta - \phi_\beta\alpha + 2V\epsilon\gamma.$$

The last member is found by expanding the first two vectors by the usual quaternion formulas, introducing ϕ, and substituting the values of $S\nabla\alpha$, $S\nabla\beta$. This form will be further reduced.

5. Multiplying $\nabla\gamma$ into γ and taking the scalar we have (since $S\lambda\phi'\mu = S\mu\phi\lambda$)

$$S\gamma\nabla\gamma = S\alpha\nabla\alpha + S\beta\nabla\beta + 2S\beta\phi_\alpha\gamma.$$

If now we set $2p = S\alpha\nabla\alpha + S\beta\nabla\beta + S\gamma\nabla\gamma$ we have

$$S\alpha\phi_\beta\gamma = -S\beta\phi_\alpha\gamma = p - S\gamma\nabla\gamma, \qquad S\beta\phi_\gamma\alpha = -S\gamma\phi_\beta\alpha = p - S\alpha\nabla\alpha,$$
$$S\gamma\phi_\alpha\beta = -S\alpha\phi_\gamma\beta = p - S\beta\nabla\beta.$$

Multiplying α into $\nabla\gamma$, taking the scalar, reducing by previous formulas, and writing the corresponding formulas cyclically,

$$S\gamma\nabla\alpha = S\beta\phi_\alpha\alpha = -S\alpha\phi_\beta\alpha = c_\alpha S\beta\nu_\alpha,$$
$$S\alpha\nabla\beta = S\gamma\phi_\beta\beta = -S\beta\phi_\gamma\beta = c_\beta S\gamma\nu_\beta,$$
$$S\beta\nabla\gamma = S\alpha\phi_\gamma\gamma = -S\gamma\phi_\alpha\gamma = c_\gamma S\alpha\nu_\gamma.$$

Multiplying by β and taking the scalar, and writing corresponding formulas:

$$S\gamma\nabla\beta = S\beta\phi_\alpha\beta = -S\alpha\phi_\beta\beta = -c_\beta S\alpha\nu_\beta,$$
$$S\alpha\nabla\gamma = S\gamma\phi_\beta\gamma = -S\beta\phi_\gamma\gamma = -c_\gamma S\beta\nu_\gamma,$$
$$S\beta\nabla\alpha = S\alpha\phi_\gamma\alpha = -S\gamma\phi_\alpha\alpha = -c_\alpha S\gamma\nu_\alpha.$$

6. From these results it is easy to see that if ν is the principal normal of the α curves and μ the binormal, c the curvature and t the torsion, $p_{\nu\mu}$ referring to α, ν, μ,

$$c = -S\mu\nabla\alpha = TV\alpha V\nabla\alpha, \qquad t = p_{\nu\mu} - S\alpha\nabla\alpha, \qquad c^\nu = -V\alpha V\nabla\alpha,$$
$$\nu = -UV\alpha V\nabla\alpha, \qquad \mu = -UV\alpha V\alpha V\nabla\alpha, \qquad \omega_\alpha = p_{\nu\mu}\alpha + V\nabla\alpha,$$
$$c\mu = -V\alpha V\alpha V\nabla\alpha = \alpha S\alpha\nabla\alpha + V\nabla\alpha, \qquad t = -S\alpha\nabla\alpha - S\frac{(S\alpha\nabla)V\nabla\alpha}{V\alpha V\nabla\alpha}.$$

Since the determination of the normal and the curvature depends only upon an expression of the form $\phi_\eta \eta$ for any unit vector η, similar expressions for the curves of β, γ, ν, and μ may be written down at once except for t. The determination of t depends upon the differentiation of the unit normal in a direction perpendicular to it, and consequently cannot be found as simply in the general case. However in certain special cases to be studied, the determination can be simplified. This difficulty of determination is also evident when we notice that the expression for t contains p, which depends upon the curl of β and γ as well as on that of α. In general the quantity p depends upon what β and γ are, and in the formula for t it is understood that in finding p, β, and γ must be the normal and binormal respectively.

In case β and γ are not the normal and the binormal, let the angle from the normal to β be w, then

$$\begin{aligned} \beta &= \cos w \cdot \nu + \sin w \cdot \mu, \\ \nabla\beta &= \cos w \cdot \nabla\nu + \sin w \cdot \nabla\mu + \nabla_1 w_1 \cdot \gamma, \\ \gamma &= -\sin w \cdot \nu + \cos w \cdot \mu, \\ \nabla\gamma &= -\sin w \cdot \nabla\nu + \cos w \cdot \nabla\mu - \nabla_1 w_1 \cdot \beta. \end{aligned}$$

Hence we have

$$S\beta\nabla\beta + S\gamma\nabla\gamma = S\nu\nabla\nu + S\mu\nabla\mu - 2S\alpha\nabla w,$$

and referred to α, β, γ we have for the value of p in terms of $p_{\nu\mu}$ (the value of p in terms of α, ν, μ)

$$2p = S\alpha\nabla\alpha + S\beta\nabla\beta + S\gamma\nabla\gamma = 2p_{\nu\mu} - 2S\alpha\nabla w,$$

or $p = p_{\nu\mu} +$ derivative of w for displacements in direction α. That is to say the quantity p belonging to arbitrary axes will be the corresponding quantity for the intrinsic axes of the curve plus the angular rate of rotation of the arbitrary system β, γ about α, for displacements along α. If this angular rotation is zero, so that β maintains a fixed angle with the normal, then there is no change in p. The torsion of α in this case can be found from the α, β, γ trihedral.

In particular if we consider the β curves and let α be the principal normal for the β curve and also for the γ curve then for any β curve the torsion is $p - S\beta\nabla\beta$, and likewise for any γ curve the torsion is $p - S\gamma\nabla\gamma$. It follows that the sum

$$t_\beta + t_\gamma = 2p - S\beta\nabla\beta - S\gamma\nabla\gamma = S\alpha\nabla\alpha$$

and this is independent of the particular curves β and γ provided α is their principal normal.* Hence the sum of the torsions of any two orthogonal

* R. A. P. Rogers, *Some differential properties of the orthogonal trajectories of a congruence of curves, with an application to curl and divergence of vectors*, Proceedings of the Royal Irish Academy, 29A (1912), 92–117.

curves for which α is the principal normal is also $S\alpha\nabla\alpha$. For, the tangent to any curve of which α is the principal normal can be written $\beta \cos u + \gamma \sin u$. Hence the normal α times the curvature c is given by

$$\begin{aligned} - S(\beta \cos u + \gamma \sin u)\nabla \cdot (\beta \cos u + \gamma \sin u) \\ = \alpha(c_\beta \cos^2 u + c_\gamma \sin^2 u + \sin u \cos u)(S\gamma\phi_\alpha \beta + S\beta\phi_\alpha \gamma) \\ + \beta(\sin u \cos u(S\beta\nabla u + S\beta\phi_\gamma \beta) + \sin^2 u S\gamma\nabla u) \\ - \gamma(\sin u \cos u(S\gamma\nabla u - S\gamma\phi_\beta \gamma) + \cos^2 u S\beta\nabla u). \end{aligned}$$

The terms in β and γ must however vanish by hypothesis so that in as much as u is an arbitrary angle we must have along the curve in question $\nabla u = 0$, since $S\beta\phi_\gamma \beta = - S\gamma\phi_\beta \beta = S\gamma\alpha = 0$, and $S\gamma\phi_\beta \gamma = - S\beta\phi_\gamma \gamma = 0$. Hence along any curve for which the vector α is the principal normal the tangent will make fixed angles with β and γ, or what is the same thing the trihedral maintains a constant relation to the fundamental trihedral. Hence for any two such curves with perpendicular tangents, the sum of the torsions is $S\alpha\nabla\alpha$.*

It follows that the quantity p is the sum of the mean torsions of curves normal to each of the three vectors α, β, γ, respectively. These curves cannot be taken so as to serve for different principal normals, unless they are straight lines. For instance the two used for α as principal normal cannot be used also for β as principal normal.

Further the curvature of a curve with α as principal normal is

$$c_\beta \cos^2 u + c_\gamma \sin^2 u + \sin u \cos u[S\beta(\phi_\alpha + \phi_\alpha')\gamma].$$

If we add to this the curvature of a perpendicular curve of the set we have $c_\beta + c_\gamma$, which does not contain u, and is the same therefore for all such perpendicular pairs; it reduces to $S\beta\phi_\alpha \beta + S\gamma\phi_\alpha \gamma$, that is to $- m_1(\phi_\alpha)$ where $m_1(\phi)$ is the first scalar invariant of the operator ϕ, and for ϕ_α is $+ S\nabla\alpha$. Hence the sum of the curvatures of the perpendicular pair of normal curves is $- S\nabla\alpha$.*

Since $S\beta\nabla\alpha$ is the projection of the vector curvature of α on γ, the perpendicular to both α and β, and $S\gamma\nabla\alpha$ the projection of this vector curvature on $-\beta$, the common perpendicular to γ and α, we see that the curl of α consists of a vector along α and a vector in the plane perpendicular to α, the latter being the numerical curvature times the unit vector which is the binormal. Hence the components of β along the unit normal and the unit binormal are $- S\gamma\nabla\alpha/c$ and $- S\beta\nabla\alpha/c$, and the corresponding components of γ are $S\beta\nabla\alpha/c$ and $- S\gamma\nabla\alpha/c$. In case $V\alpha V\nabla\alpha = 0$ these components

* R. A. P. Rogers, *Some differential properties of the orthogonal trajectories of a congruence of curves, with an application to curl and divergence of vectors*, Proceedings of the Royal Irish Academy, 29A (1912), 92–117.

become indeterminate. In case $S\alpha\nabla\alpha = 0$, α is perpendicular to $V\nabla\alpha$ and c becomes merely $TV\nabla\alpha$. Also we notice that in any case

$$p = t_a + S\alpha\nabla\alpha + dw/ds.$$

THE OPERATORS ϕ_a, ϕ_β, ϕ_γ

7. Since $\phi'_a\,\alpha = 0$, the transverse of ϕ_a has at least one zero root, and therefore ϕ_a has at least one zero root. The form of ϕ_a must then be either

(a) $-\beta S\zeta_3 + \gamma S\zeta_2$, $\zeta_3 \neq 0$, $\zeta_2 \neq 0$, $V\zeta_3\,\zeta_2 \neq 0$,

(b) $\beta' S\zeta$, $\zeta \neq 0$, where β' is perpendicular to α, $\beta'^2 = -1$,

(c) 0.

In case (c) α is constant and its congruence consists of parallel straight lines. The β and γ curves lie in parallel planes perpendicular to α, that is α is their common binormal. This is evident also from the Serret-Frenet formulas. For we have

$$\phi_a = -S(\,)\,\nabla\cdot\alpha = -S(\,)\,\nabla\cdot\beta\gamma = V\beta\phi_\gamma - V\gamma\phi_\beta,$$

and if this vanishes its transverse vanishes, that is $\phi'_\beta\,V\gamma - \phi'_\gamma\,V\beta = 0$, whence

$$\phi'_\beta\,\alpha = 0, \qquad \phi'_\gamma\,\alpha = 0, \qquad \text{and as} \qquad \phi'_\beta\,\beta = 0, \qquad \phi'_\gamma\,\gamma = 0,$$

$$\phi_\beta = \gamma S\xi, \qquad \phi_\gamma = -\beta S\xi, \qquad \phi_\beta\,\beta = \gamma S\xi\beta, \qquad \phi_\gamma\,\gamma = -\beta S\xi\gamma,$$

so that the binormals are α in each case. The β curves in their plane are respectively orthogonal to the γ curves in the same plane. The curvature of the β curves is the projection of ξ on their normal, and likewise for the curvature of the γ curves. The projection of ξ on α is the quantity $-p$.

In case (b) we find that for some vector τ

$$\phi_\beta = -\alpha S\zeta\cos u + \gamma S\tau, \qquad \phi_\gamma = -\alpha S\zeta\sin u - \beta S\tau,$$

where $\beta' = \beta\cos u + \gamma\sin u$. In this case $d\alpha = +\beta' S\zeta d\rho$ for any $d\rho$. But if $d\rho$ is taken along the α curve $d\alpha = c_a\,\nu_a$ hence $+\beta' = \nu_a$, $\alpha\beta = \mu_a$, and $+S\alpha\zeta = c_a$. We have at once since $m_1(\phi_a) = S\nabla\alpha$ etc.

$$S\nabla\alpha = S\beta'\,\zeta, \qquad S\nabla\beta = -\cos u\,S\alpha\zeta + S\gamma\tau, \qquad S\nabla\gamma = -\sin u S\alpha\zeta - S\beta\tau,$$

and

$$2\epsilon = V\zeta V\alpha\beta' + V\tau\alpha = V\zeta\mu_a + V\tau\alpha.$$

Again since the double spin vector of ϕ_a is $V\nabla\alpha$,

$$V\nabla\alpha = V\beta'\,\zeta, \qquad V\nabla\beta = -V\alpha\zeta\cos u + V\gamma\tau,$$

$$V\nabla\gamma = -V\alpha\zeta\sin u - V\beta\tau,$$

$$S\alpha\nabla\alpha = S\mu_a\zeta, \qquad S\beta\nabla\beta = S\gamma\zeta \cos u + S\alpha\tau,$$

$$S\gamma\nabla\gamma = -S\beta\zeta \sin u + S\alpha\tau,$$

$$2p = 2S\mu_a\zeta + 2S\alpha\tau, \qquad \text{or} \qquad p = S\mu_a\zeta + S\alpha\tau.$$

Differentiating the normal along α, we find from the Frenet formulas

$$t_a = S\tau\alpha - S\alpha\nabla u, \qquad \omega_a = [\alpha S\alpha(\tau - \nabla u) - S\alpha\zeta]\beta'.$$

If β is the normal ν_a, and γ the binormal μ_a, $S\alpha\tau$ is the torsion of the α curves.

When $d\rho$ is in the plane $S\zeta\pi = 0$, $d\alpha = 0$, and hence for all infinitesimal displacements of the vertex of the trihedral of α, β, γ in a plane perpendicular to ζ, α remains constant, $d\beta$ becomes $\gamma S\tau d\rho$ or $V\alpha\beta S\tau d\rho$ and $d\gamma$ is

$$-\beta S\tau d\rho = V\alpha\gamma S\tau d\rho.$$

That is to say, the trihedral is rotated about α by the angular rotation $S\tau d\rho$. Hence for congruences orthogonal to the congruence $Vd\rho\zeta = 0$, the trihedral merely rotates about α by an amount equal to the projection of τ on the tangent of the curve. In case then the ζ congruence is a normal congruence, the normal surfaces are such that displacements of the vertex on any such surface are accompanied by rotation about α. If $d\rho$ is perpendicular to τ there is no rotation, so that for displacements along the congruence $Vd\rho V\tau\zeta = 0$ there is no rotation of the trihedral at all. Particular cases would occur if $V\zeta\tau = 0$ or if $\tau = 0$.

Since we can write $d\alpha = -V\alpha\mu_a S\zeta d\rho$, $d\beta = -V\beta(\mu_a S\zeta d\rho + \alpha S\tau d\rho)$, $d\gamma = -V\gamma(\mu_a S\zeta d\rho + \alpha S\tau d\rho)$, it is evident that for any case whatever of displacement $d\rho$, there is a rotation given by the vector $+\mu_a S\zeta d\rho + \alpha S\tau d\rho$. In case $\tau = 0$ there is rotation about the binormal of the α curve and α is the common normal of the β and the γ curves. In any case there is rotation about a line in the rectifying plane of α, the vector axis of rotation being a linear function $\theta = \mu_a S\zeta + \alpha S\tau$ of the displacement. This function is such that its transverse has the normal of the α curves for a zero axis. It is also clear that if ϕ_a is of nullity* two, then ϕ_β and ϕ_γ are of nullity unity or nullity two together, (since they depend directly upon the same vectors ζ, τ) unless it happen that β' is $= \beta$ or $= \gamma$ in which case ϕ_γ in the first instance, ϕ_β in the second, is of nullity one greater than that of the other.

In case (a) it is not difficult to find that the general forms of $\phi_\beta, \phi_\gamma, \phi_a$ are:

$$\phi_\beta = -\gamma S\zeta_1 + \alpha S\zeta_3, \qquad \phi_\gamma = -\alpha S\zeta_2 + \beta S\zeta_1, \qquad \phi_a = -\beta S\zeta_3 + \gamma S\zeta_2.$$

These follow since $\phi'_\beta\gamma = -\phi'_\gamma\beta$ etc. It is evident that when ϕ_a is given we also know ϕ_β and ϕ_γ when further a single vector ζ_1 is given or found. Since in this case ζ_2, ζ_3 are not zero and not parallel, ϕ_a is of nullity unity.

* Nullity equals order minus rank, that is, here, nullity two and rank one mean the same.

We may have however ζ_1 either zero or parallel to ζ_2 or ζ_3. So that ϕ_β or ϕ_γ or both may have nullity two but not nullity three. The invariant axis of ϕ_a for the root 0 is easily found, for the adjunct operator ψ'_a is

$$V\beta\gamma SV\zeta_3\,\zeta_2 = -\,\alpha SV\zeta_2\,\zeta_3$$

so that the invariant axis $V\zeta_2\,\zeta_3 = \psi_a\,\alpha = \frac{1}{2}\,V\nabla_1\,\nabla_2\,S\alpha\alpha_1\,\alpha_2$. This may also be easily found by noticing that $-\,\zeta_2 = \phi'_a\,\gamma$, or $\zeta_2 = \nabla_1\,S\alpha_1\,\gamma$, and $\zeta_3 = -\,\nabla_1\,S\alpha_1\,\beta$.

We may have different possibilities for the other axes and roots of ϕ_a. We may have another zero root, in which case the invariant $m_2 = 0$, that is $SV\nabla_1\,\nabla_2\,V\alpha_1\,\alpha_2 = 0$, or in terms of ζ_2, ζ_3, $S\alpha\zeta_2\,\zeta_3 = 0$ and α, ζ_2, and ζ_3 are in one plane. Hence $V\alpha\zeta_2$, $V\alpha\zeta_3$, $V\zeta_2\,\zeta_3$ are in the same direction β'' in the plane of β and γ.

If now the nullity is to remain unity, ϕ_a must convert some other direction into that of the invariant direction β''. This new direction cannot be in the plane of β and γ, for if ϕ_a converts this direction into β'' and annuls β'', ϕ_a^2 would annul the whole β, γ plane, and as ϕ_a converts all vectors into this plane, ϕ_a^3 would annul all vectors and there would be three zero roots. Hence with just two zero roots and nullity unity there is a direction ξ not in the β, γ plane which is converted into β'' by ϕ_a. Now if $\phi'_a\,\zeta_2$ is not parallel to β'', $V\beta''\,\phi'_a\,\zeta_2$ is such a vector. For $\phi_a\,V\beta''\,\phi'_a\,\zeta_2$ is parallel to $\phi_a\,V\phi'_a\,\zeta_2\,V\alpha\zeta_2$ or $\phi_a\,\zeta_2\,S\zeta_2\,\phi_a\,\alpha - \phi_a\,\alpha S\zeta_2\,\phi_a\,\zeta_2$ or $V\zeta_2\,V\phi_a\,\alpha\phi_a\,\zeta_2$ or finally $V\zeta_2\,\alpha$, hence is parallel to β''. Now ϕ'_a converts vectors into the plane of ζ_2 and ζ_3 hence they cannot be parallel to β'', perpendicular to this plane. However $\phi'_a\,\zeta_2$ might vanish on account of ζ_2 being parallel to α. In such case ζ_3 is not parallel to α and will answer just as well to determine the direction in question. Hence we have ξ a unit vector in the direction of $V\beta''\,\phi'_a\,\zeta_2$ or $V\beta''\,\phi'_a\,\zeta_3$. This is clearly perpendicular to $\phi'_a\,\zeta_2$, and to β'', hence is the line in the plane of α, ζ_2, ζ_3 which is perpendicular to $\phi'_a\,\zeta_2$. But $\phi'_a\,\alpha = 0$, hence if β' is the intersection of the plane of α, ζ_2, ζ_3 with the β, γ plane, ϕ'_a will give for any vector in this plane a multiple of $\phi'_a\,\beta'$ so that if $\beta' = \beta\cos u + \gamma\sin u$, the direction ξ is in the plane and is perpendicular to $-\,\zeta_3\cos u + \zeta_2\sin u$. It may be written $V\beta''(-\,\zeta_3\cos u + \zeta_2\sin u)$. Now there must under the hypotheses be one invariant axis in the β, γ plane not in the direction of β'', and with a root not zero; let such direction be δ. Then

$$\phi_a = -\,g\delta S\delta' + h\beta''\,SV\beta''\,\beta'$$

where δ' is in plane of β, γ and $= V\alpha\beta''$, hence

$$\phi_a = (-\,g\delta S\alpha + h\beta''\,S\beta')\,V\beta'', \qquad \text{and} \qquad \phi'_a = V\beta''\,(g\alpha S\delta - h\beta'\,S\beta''),$$

$$\zeta_2 = -\,V\beta''\,(g\alpha S\delta\gamma - h\beta'\,S\beta''\,\gamma),$$

$$\zeta_3 = V\beta''\,(g\alpha S\delta\beta - h\beta'\,S\beta''\,\beta).$$

THE OPERATOR θ

8. It will not have escaped attention that the three operators ϕ_α, ϕ_β, ϕ_γ are intimately connected and have elements in common, as for instance the vectors ζ_1, ζ_2, ζ_3. In fact it is evident at once that if we let

$$\theta = -\alpha S\zeta_1 - \beta S\zeta_2 - \gamma S\zeta_3,$$

then

$$\phi_\alpha = -V\alpha\theta, \qquad \phi_\beta = -V\beta\theta, \qquad \phi_\gamma = -V\gamma\theta,$$

and that we can also write (since $S\alpha\phi_\alpha = 0$, etc.)

$$2\theta = \alpha\phi_\alpha + \beta\phi_\beta + \gamma\phi_\gamma.$$

We may arrive at θ from another starting point however. If we consider the trihedral of the three unit vectors α, β, γ moving from one position to another, the original position having been α_0, β_0, γ_0, the position α, β, γ could be produced by a rotation $q(\ )\bar{q}$, where q is a unit quaternion, and $\bar{q}$ is the conjugate unit quaternion, that is

$$\alpha = q\alpha_0\bar{q}, \qquad \beta = q\beta_0\bar{q}, \qquad \gamma = q\gamma_0\bar{q}.$$

The axis of q is the axis of rotation, and the angle of q is half the angle of rotation. It is important to notice that q is a definite function of α, β, γ.*

If now we displace the vertex by an amount $d\rho$, the differential change in the rotation would be given by $2Vdq\bar{q}$ or $2dq\bar{q}$, since Tq is 1 and $Sdq\bar{q} = 0$. This is a linear vector function of $d\rho$, say $\theta d\rho$, that is,

$$\theta d\rho = -2Sd\rho\nabla_1 \cdot q_1\bar{q}.$$

The displacement of the end of α would be $V\theta(d\rho)\alpha = d\alpha$, so that $\phi_\alpha = -V\alpha\theta$, and so for the others. From this it would follow at once that

$$\theta = \alpha\phi_\alpha + \alpha \text{ times a linear scalar function} = \alpha\phi_\alpha - \alpha S\zeta_1.$$

We can easily now arrive at the form given above. We notice that for any $\eta = a\alpha + b\beta + c\gamma$, $T\eta = 1$, a, b, c constant, we must have $\theta = \eta\phi_\eta - \eta S\zeta$. This prevents θ from being arbitrary absolutely.

9. If for q we substitute qa, we see at once that if a is a constant quaternion θ is not changed. That is we may use for α, β, γ any other trihedral of mutually orthogonal unit vectors, which maintain constant angles with α, β, γ in all positions. If however a is not constant then

$$\theta_1 = \theta + q\omega\bar{q}, \qquad \text{where} \qquad \omega = -2S(\ )\nabla_1 \cdot a_1\bar{a}.$$

θ is the operator that converts a displacement $d\rho$ into the resulting instantaneous rotation, represented by an axis and a length on it which measures

* See Joly, *Manual of Quaternions*, p. 26, Ex. 6.

the rotation rate; that is, for a displacement in the direction η, the rotation has the direction and rate of rotation given by $\theta\eta$. When this is multiplied by ds, the length of the displacement, it gives the instantaneous rotation. It might be called a *rotation derivative*.

10. By using a fundamental identity of quaternions we have

$$-(\ )\, S\alpha\nabla\alpha = V\alpha\alpha_1 S\nabla_1 + V\alpha_1 \nabla_1 S\alpha + V\nabla_1 \alpha S\alpha_1 .$$

Adding similar forms for β and γ we have

$$(\ )\,[S\alpha\nabla\alpha + S\beta\nabla\beta + S\gamma\nabla\gamma] = \alpha\phi_\alpha + \beta\phi_\beta + \gamma\phi_\gamma - V\nabla_1 \alpha S\alpha_1$$
$$- V\nabla_1 \beta S\beta_1 - V\nabla_1 \gamma S\gamma_1 - V\alpha_1 \nabla_1 S\alpha - V\beta_1 \nabla_1 S\beta - V\gamma_1 \nabla_1 S\gamma .$$

But we have always $(\ ) = -\alpha S\alpha - \beta S\beta - \gamma S\gamma$ so that operating on this with ∇, and transposing,

$$V\nabla_1 \alpha S\alpha_1 + V\nabla_1 \beta S\beta_1 + V\nabla_1 \gamma S\gamma_1 = -V\nabla_1 \alpha_1 S\alpha - V\nabla_1 \beta_1 S\beta - V\nabla_1 \gamma_1 S\gamma .$$

Hence if we set, as in § 5,

$$2p = S\alpha\nabla\alpha + S\beta\nabla\beta + S\gamma\nabla\gamma$$

we find from the above the important formula

$$2p\,(\ ) = 2\theta + 2V\nabla_1 \alpha_1 S\alpha + 2V\nabla_1 \beta_1 S\beta + 2V\nabla_1 \gamma_1 S\gamma ,$$

or transposing and limiting ∇ to the first following vector,

$$\theta = p - V\nabla\alpha S\alpha - V\nabla\beta S\beta - V\nabla\gamma S\gamma ,$$

which gives θ in terms of the *curls* of α, β, γ.

This may also be written

$$\theta = p - V\alpha\nabla S\alpha - V\beta\nabla S\beta - V\gamma\nabla S\gamma ,$$

where ∇ acts on the following vector, and this reduces at once to

$$p + V\alpha\phi_\alpha' + V\beta\phi_\beta' + V\gamma\phi_\gamma' .$$

Whence we have

$$\theta' = p - \phi_\alpha V\alpha - \phi_\beta V\beta - \phi_\gamma V\gamma .$$

We now have

$$\zeta_1 = p\alpha + \phi_\beta \gamma - \phi_\gamma \beta , \quad \zeta_2 = p\beta + \phi_\gamma \alpha - \phi_\alpha \gamma , \quad \zeta_3 = p\gamma + \phi_\alpha \beta - \phi_\beta \alpha .$$

Since $\phi_\beta = V\gamma\phi_\alpha - V\alpha\phi_\gamma$, etc., these reduce to

$$\zeta_1 = p\alpha + V\nabla\alpha + 2V\alpha\epsilon , \qquad \zeta_2 = p\beta + V\nabla\beta + 2V\beta\epsilon ,$$
$$\zeta_3 = p\gamma + V\nabla\gamma + 2V\gamma\epsilon .$$

These determine the vectors ζ_1, ζ_2, ζ_3 in terms of α, β, γ respectively, since ϵ

is already known in terms of α, β, γ. If η preserves invariable angles with α, β, γ,

$$\theta\eta = p\eta + V\nabla\eta, \qquad \theta'\eta = p\eta + V\nabla\eta + 2V\eta\epsilon.$$

11. The first scalar invariant of θ, designated by m_1 is

$$3p - S\alpha\nabla\alpha - S\beta\nabla\beta - S\gamma\nabla\gamma = p.$$

From this we can form the function

$$\chi'_\theta = m_1 - \theta = V\nabla_1\alpha_1 S\alpha + V\nabla_1\beta_1 S\beta + V\nabla_1\gamma_1 S\gamma.$$

It is obvious at once that the curls are given in terms of θ in the form

$$V\nabla\alpha = -\chi'_\theta\alpha, \qquad \text{etc.}$$

The vector called the spin-vector of θ, is the same as ϵ in § 3,

$$\epsilon_\theta = \tfrac{1}{2}[V\alpha V\nabla\alpha + V\beta V\nabla\beta + V\gamma V\nabla\gamma]$$
$$= -\tfrac{1}{2}(\phi_\alpha\alpha + \phi_\beta\beta + \phi_\gamma\gamma) = -\tfrac{1}{2}[\alpha S\nabla\alpha + \beta S\nabla\beta + \gamma S\nabla\gamma].$$

A necessary and sufficient condition that $\epsilon = 0$ is that the three convergences vanish. If the curls vanish, $\epsilon = 0$, and the divergences vanish, which we would also know from the equations $S\nabla\alpha = S\gamma\nabla\beta - S\beta\nabla\gamma$, etc., so that the vanishing of the curls is a sufficient condition, for the vanishing of ϵ. However as p also vanishes, $\theta = 0$ in this case whereas the vanishing of ϵ means that $\theta = \theta'$, that is, θ is self-transverse. Hence the vanishing of the curls is not a necessary condition that $\epsilon = 0$.

When $\theta = 0$ each of the operators $\phi_\alpha, \phi_\beta, \phi_\gamma$ vanishes, and α, β, γ are constant. When θ is self-transverse the three sets of curves which have as principal normals the three vectors α, β, γ respectively, are such that perpendicular curves of the same set have equal curvatures and one normal is opposite to the other; that is, taking the normal curves of α for instance, if one is concave in the direction of α the perpendicular curve is concave in the direction of $-\alpha$.

If we subtract the last form of ϵ from the first we have

$$V\alpha\nabla\alpha + V\beta\nabla\beta + V\gamma\nabla\gamma = 0.$$

Adding $2p$ we have

$$\alpha\nabla\alpha + \beta\nabla\beta + \gamma\nabla\gamma = 2p.$$

12. Since $\theta\alpha = V\nabla\alpha + p\alpha$, it is clear that if $V\nabla\alpha = 0$ then α is an invariant axis of θ, and since $\phi_\alpha\alpha = 0$, the α curves are straight. Hence $V\nabla\alpha = 0$ is a sufficient condition that the α curves be straight. But conversely if α is an invariant axis, either $V\nabla\alpha = 0$ or else $V\alpha V\nabla\alpha = 0 = \phi_\alpha\alpha$ hence the curvature of the α curves is zero, and they are straight lines. The necessary and sufficient condition is therefore $V\cdot\alpha V\nabla\alpha = 0$. If η is any vector which

maintains constant angles with α, β, γ then $\theta\eta = V\nabla\eta + p\eta$, and if η is an invariant axis for θ at all points, so that $V\eta\theta\eta = 0 = V\eta V\nabla\eta$, then the η congruence consists of straight lines, and displacements along them would be accompanied by rotation about them. Since $\theta\alpha$ is the instantaneous rotation that accompanies a displacement along α, the component of this rotation along α is $-\alpha S\alpha\theta\alpha$. In case β is the principal normal of the α curve and γ its binormal, for all positions, then $-S\alpha\theta\alpha$ is the rate of rotation of the osculating plane and is therefore the torsion of the α curve. However it is to be remembered that in this expression of the torsion θ is dependent upon the normal and binormal. θ does not change however if for the normal and the binormal we substitute β', γ' two perpendicular unit vectors in the plane of β, γ which maintain constant angles with them. Since $-S\alpha\theta\alpha = p - S\alpha\nabla\alpha$ we see that when α is the principal normal of the β curves and of the γ curves we have

$$- S\alpha\theta\alpha = p - t_\beta - t_\gamma .$$

This relation will hold if β and γ are the tangents of any two perpendicular normal curves of α. We may write this in the form

$$t_\beta + t_\gamma = p + S\alpha\theta\alpha .$$

In case α is the direction of the principal normal of a curve whose tangent is β we must have as the necessary and sufficient condition

$$V\alpha\phi_\beta\,\beta = 0 = - V\alpha V\beta\theta\beta = \beta S\alpha\theta\beta = S\alpha\theta\beta = S\beta\theta'\,\alpha .$$

Likewise if it has the direction of the principal normal of a γ curve we must have $S\alpha\theta\gamma = 0$, hence $S\gamma\theta'\,\alpha = 0$ also. Under these conditions

$$V\theta'\,\alpha V\beta\gamma = 0 \qquad \text{or} \qquad V\alpha\theta'\,\alpha = 0 .$$

Hence the necessary and sufficient condition that α is in the direction of the principal normal of the β curves and the γ curves, and hence of all curves whose tangents maintain constant angles with β and γ, is

$$V\alpha\theta'\,\alpha = 0 .$$

13. Since $2\theta = -\alpha\alpha_1 S\nabla_1 - \beta\beta_1 S\nabla_1 - \gamma\gamma_1 S\nabla_1$ we find at once that

$$2\theta V\nabla = - V\alpha_2\,\alpha_1 S\nabla_1\nabla_2 - VV\beta_2\,\beta_1 S\nabla_1\nabla_2 - V\gamma_2\,\gamma_1 S\nabla_1\nabla_2 ,$$

where ∇ on the left operates only on θ and where each term on the right contains a vacant place for the operand of the linear vector operator. But this is the same as

$$\theta V\nabla = - \psi'_\alpha - \psi'_\beta - \psi'_\gamma .$$

But since $\phi_\alpha = - V\alpha\theta$ we have for all λ, μ,

$$\psi'_\alpha\,V\lambda\mu = V\phi_\alpha\lambda\phi_\alpha\,\mu = VV\alpha\theta\lambda V\alpha\theta\mu = - \alpha S\alpha V\theta\lambda\theta\mu = - \alpha S\alpha\psi'_\theta\,V\lambda\mu$$

so that $\psi_a' = -\alpha S\alpha\psi_\theta'$ with similar forms for the others. Substituting these we have

$$\psi_a' = -\theta V\nabla.$$

This is an important form. When it vanishes identically we have the necessary and sufficient condition fulfilled that $\theta d\rho$ be an exact differential. In this case we could write $\theta d\rho = -Sd\rho\nabla\cdot\sigma$ where σ is properly chosen. That is, the form of θ can be reduced to $\theta = -S(\,)\nabla\cdot\sigma$. Hence $p = S\nabla\sigma$, $2\epsilon = V\nabla\sigma$, $S\nabla\alpha = S\alpha\nabla\sigma$, $S\nabla\beta = S\beta\nabla\sigma$, $S\nabla\gamma = S\gamma\nabla\sigma$, $c_a\nu_a = -V\alpha\phi_\sigma\alpha$, etc. This particular case leads to some very interesting possibilities.

From the equation above we have

$$\psi_\theta = V\nabla_1\theta_1'.$$

From this it is easy to arrive at the second scalar invariant of θ, for it is the first scalar invariant of ψ_θ, and it is easy to prove that for the general case of an operator like this the first scalar invariant is $-2S\nabla\epsilon$.

Hence $m_2 = -2S\nabla\epsilon$. This can be verified by the direct calculation from the first or other form of θ.

Since $\psi_\theta\theta = m_3(\,)$, we have taking the first scalar invariant of this

$$3m_3 = 2S\nabla_1\epsilon_{\theta_1'\theta},$$

where the subscript is removed after the differentiations. When $m_3 = 0$, θ has at least one zero root. In case ψ_θ is not also zero nor m_2 zero, there is but one zero root. The invariant axis of this root is $\psi_\theta\delta$ where δ is any unit vector. If also $m_2 = 0$ but ψ_θ is not zero, zero is a repeated root with one invariant axis, but θ^2 has a plane of invariant axes. The other root is then $m_1 = p$. If $m_3 = 0$, $\psi_\theta = 0$, since there must be a repeated root, we must also have $m_2 = 0$, and the zero root has two invariant axes, and hence a whole plane of invariant axes. In case all the scalar invariants are zero, there is a triple zero root with one invariant axis, θ^2 has an invariant plane, and θ^3 annuls all vectors. If $\theta^2 = 0$, there is a triple zero root, with an infinity of axes in a plane. If $\theta = 0$ any line is an axis.

14. Other expressions for the scalar invariants m_2 and m_3 are as follows:

$$m_2 = -p^2 - S\alpha V\nabla\beta V\nabla\gamma - S\beta V\nabla\gamma V\nabla\alpha - S\gamma V\nabla\alpha V\nabla\beta,$$

$$m_3 = pm_2 - SV\nabla\alpha V\nabla\beta V\nabla\gamma.$$

We also have

$$2\epsilon + 2p = \alpha V\nabla\alpha + \beta V\nabla\beta + \gamma V\nabla\gamma, \quad 4\epsilon = VV\nabla\alpha\alpha + VV\nabla\beta\beta + VV\nabla\gamma\gamma,$$

$$2p - 2\epsilon = V\nabla\alpha\alpha + V\nabla\beta\beta + V\nabla\gamma\gamma, \quad 0 = V\alpha\nabla\alpha + V\beta\nabla\beta + V\gamma\nabla\gamma,$$

$$\theta(\epsilon) = p\epsilon - \frac{V\nabla\alpha S\nabla\alpha - V\nabla\beta S\nabla\beta - V\nabla\gamma S\nabla\gamma}{2} = -\epsilon_{\psi_\theta} = \epsilon_{\psi_{\theta'}}$$

$$= p\epsilon - \tfrac{1}{2}V\alpha VV\nabla\beta V\nabla\gamma - \tfrac{1}{2}V\beta VV\nabla\gamma V\nabla\alpha - \tfrac{1}{2}V\gamma VV\nabla\alpha V\nabla\beta,$$

$$S\epsilon\theta\epsilon = -S\epsilon\epsilon_{\psi_a}.$$

In all the above formulas ∇ operates only on the first following symbol.

Another class of formulas which are useful are the following:

$$
\begin{aligned}
\nabla S \nabla \alpha = 2\nabla S \alpha \epsilon &= -2\phi'_\alpha \epsilon + 2\nabla_1 S \epsilon_1 \alpha \\
&= 2\theta' V \epsilon \alpha + 2\nabla_1 S \epsilon_1 \alpha = \theta' \theta \alpha - \theta'^2 \alpha - 2\phi'_\epsilon \alpha,
\end{aligned}
$$

$$
S\alpha \nabla S \nabla \alpha + S\beta \nabla S \nabla \beta + S\gamma \nabla S \nabla \gamma = 4\epsilon^2 - 2S\nabla\epsilon,
$$

$$
\begin{aligned}
V\alpha \nabla S \nabla \alpha &+ V\beta \nabla S \nabla \beta + V\gamma \nabla S \nabla \gamma \\
&= -2V\alpha\theta' V\alpha\epsilon - \text{etc.} + 2V\alpha(\nabla_1 S\epsilon_1)\alpha + \text{etc.} \\
&= -2V\nabla\beta\gamma\theta' V\alpha\epsilon - \text{etc.} + \text{etc.} \\
&= 2\theta\epsilon - 2p\epsilon - 4V\nabla\epsilon.
\end{aligned}
$$

Numerous other relations can be written down easily.

15. The linear vector function θ has at least one invariant axis, say ξ_1. This axis is such that $\theta\xi_1 = g_1 \xi_1$. There is therefore at every point one direction at least such that displacements in that direction are accompanied by rotation about the same direction, though the rate g_1 may be zero. Consequently these directions determine at least one congruence of curves such that if the vertex of the trihedral travels on any one of the curves the trihedral will rotate about the tangent of the curve, the amount of rotation depending upon the position. The motion of the trihedral is a sort of screw motion. In case the roots of θ are all distinct, they have shear regions of one dimension, and there will be at each point three curves, the whole constituting three congruences, such that displacement along any one of the curves is accompanied by a screw motion along the curve. Since any point can be reached by displacements on the three congruences, any displacement and consequent rotations of the trihedral can be analyzed into three successive screw motions, If one of the distinct roots is zero, the rotation about the axis is of zero magnitude and the displacement is accompanied by a mere gliding, the trihedral remaining parallel to itself. If there are two zero roots which have distinct invariant axes, that is if they have shear regions of one dimension each, they may be any two distinct directions in the plane of the two shear regions. In such case the plane containing all the invariant axes would envelope a surface such that displacements from any point of the surface to any other point would be accompanied by no rotation whatever. If we have the similar case for all three roots, $\theta = 0$ and any motion is a mere translation.

When one of the roots is repeated, even a zero root, but the shear region is of two dimensions, so that there is only one invariant axis, then θ converts any other definite direction in the shear region into a multiple of itself, which may be zero, plus a multiple of the unit vector of the invariant axis. Hence

at each point there would be one tangent line of a curve belonging to a congruence along which the motion is a screw motion on the curve, and there is also a tangent to a curve for which displacements are screw motions along the curve and also a rotation about a straight line which would be in the direction of the tangent to the curve at the point belonging to the first mentioned congruence. In case this double root is zero, there will be a congruence for which the trihedral simply moves parallel to itself, and there will also be a congruence, for which the trihedral moves parallel to itself with a superimposed rotation at each point about the tangent of the curve at the point which belongs to the first congruence.

When there is a triple root we may have the preceding case with a third congruence for which the motion is a screw motion on the curve, this congruence and the invariant congruence of the same root determining a surface for which displacement along any curve lying on it is accompanied by a screw motion. The third congruence cuts through these surfaces and displacements along it would be screw motions accompanied by rotation about a single tangent to a curve lying on the surface. If the root is zero the rate of rotation is zero and the motions become translations, save for the last case.

Finally if there is a triple root with a shear region of three dimensions, there is a congruence for which the motion is a screw motion, another congruence for which the motion is a screw motion accompanied by a rotation about a tangent to a curve of the first congruence, and a third congruence for which motion is a screw motion accompanied by a rotation about a tangent to a curve of the second congruence. In case the root is zero, the screw motion degenerates into a translation, but the rotations remain.

In the cases of shear region of dimensions two or three the congruences accompanied by rotations are not unique, for it is clear that if

$$\theta\xi_2 = g_1\,\xi_2 + h\xi_1, \qquad \text{and} \qquad \theta\xi_1 = g_1\,\xi_1,$$

then

$$\theta\,(\xi_2 + x\xi_1) = g_1(\xi_1 + x\xi_1) + h\xi_1,$$

and similar equations hold for a shear region of order three.

16. If θ has no zero roots, then $\psi_\theta\,\alpha$ must be an axis for ϕ_α; and similarly for β, and γ, for $\phi_\alpha\,\psi_\theta\,\alpha = -\,V\alpha\theta\psi_\theta\,\alpha = -\,V\alpha\alpha = 0$. In case θ has a zero root the corresponding invariant axis or axes are also zero axes for each ϕ_α, ϕ_β, ϕ_γ. In this case if $\psi_\theta\,\alpha$ does not vanish it is also an invariant axis for a zero root of all of the operators θ and ϕ. In case there are two zero roots of θ, then ψ_θ may also vanish identically, and there is a whole plane of invariant axes of θ and each ϕ, or ψ_θ may not vanish identically, in which case again $\psi_\theta\,\alpha$ is an invariant axis for ϕ_α etc. The case of three zero roots for θ gives $\psi_\theta = \theta^2$, and if this does not vanish identically it gives $\psi_\theta\,\alpha$ an axis of ϕ_α as before. If also however $\theta^2 = 0$, and $\theta \neq 0$, then $\theta\alpha$ is an axis for ϕ_α.

If α is an invariant axis of θ it is a zero axis of ϕ_a, etc. But in this case $V\alpha\theta\alpha = 0$, hence the curvature of curves of the α congruence is everywhere zero and they are straight lines. The condition $V\alpha\theta\alpha = 0$ is evidently the necessary and sufficient condition that the α congruence consist of straight lines. Also $V\alpha\theta'\alpha = 0$ is the condition that α is everywhere the principal normal of the β and γ curves. The form of θ in this case is special, being

$$\theta = -z\alpha S\alpha - \beta S(u\beta + v\gamma) - \gamma S(u'\beta + v'\gamma).$$

Therefore $V\epsilon\alpha = 0$ and $V\epsilon\theta\epsilon = 0$ are necessary and sufficient conditions that a congruence of straight lines be principal normals to the orthogonal congruences β and γ.

17. In case $V\alpha\theta'\alpha = 0$, that is when α is an axis of θ', α is the principal normal of the β and the γ curves so that

$$t_\beta = S\gamma\phi_a\beta = -S\beta\theta\beta, \qquad t_\gamma = -S\beta\phi_a\gamma = -S\gamma\theta\gamma.$$

From these we have

$$t_\beta - t_\gamma = S\gamma\phi_a\beta + S\beta\phi_a\gamma = S\alpha V\beta(\phi_a + \phi_a')\beta.$$

Since we already have seen that

$$t_\beta + t_a = S\alpha\nabla\alpha,$$

we have at once

$$2t_\beta = S\alpha\nabla\alpha + S\alpha V\beta(\phi_a + \phi_a')\beta, \qquad 2t_\gamma = S\alpha\nabla\alpha - S\alpha V\beta(\phi_a + \phi_a')\beta.$$

Let us hold α constant now and vary β. This torsion is an extremal if

$$0 = S\alpha Vd\beta(\phi_a + \phi_a')\beta + S\alpha V\beta(\phi_a + \phi_a')d\beta.$$

But $d\beta$ is parallel to γ, since α is fixed; hence

$$0 = -S\beta(\phi_a + \phi_a')\beta + S\gamma(\phi_\phi + \phi_a')\gamma.$$

Hence we have for a maximum or minimum

$$S\beta\phi_a\beta = S\gamma\phi_a\gamma \qquad \text{or} \qquad S\gamma\nabla\beta = -S\beta\nabla\gamma = -\tfrac{1}{2}S\nabla\alpha$$

or

$$S\gamma\theta\beta + S\beta\theta\gamma = 0.$$

This may be interpreted to read: the curvatures for the curves of extreme torsions are both equal to $-\frac{1}{2}$ the convergence of α. Hence the vectors β of extreme torsion are in the directions of equal curvature for the perpendicular vectors β, γ.

18. The curvature of a β curve is easy to find when α is the normal of all the β curves, for it is

$$-S\alpha\phi_\beta\beta = S\beta\phi_a\beta.$$

This is an extremal when $Sd\beta\phi_a\beta + S\beta\phi_a d\beta = 0$, or since $d\beta$ is parallel to γ,

$$S\beta\theta\beta = S\gamma\theta\gamma .$$

That is, the direction of curves with extreme curvature is such that the torsion equals that of the perpendicular curve and is hence $= \frac{1}{2} S\alpha\nabla\alpha$.

19. The value of the torsion of the β curve is

$$2t_\beta = S\alpha\nabla\alpha + (S\beta\theta\beta - S\gamma\theta\gamma) = S\alpha\nabla\alpha + S\alpha V\beta(\phi_a + \phi_a')\beta$$

and if we let β be a direction of extreme curvature, $S\beta\theta\beta - S\gamma\theta\gamma = 0$. Suppose β' s any other direction, and

$$\beta' = \beta \cos u + \gamma \sin u , \qquad \gamma' = \alpha\beta' = -\beta \sin u + \gamma \cos u .$$

Then

$$S\beta'\theta\beta' - S\gamma'\theta\gamma' = 2S(\beta\theta\gamma + \gamma\theta\beta) \sin 2u .$$

Hence for any torsion

$$2t_{\beta'} = S\alpha\nabla\alpha + 2S(\beta\theta\gamma + \gamma\theta\beta) \sin 2u .$$

Again the value of the curvature of the β curve is $S\gamma\theta\beta$, and if we measure u now from the direction of extreme torsion, we have

$$S\gamma'\theta\beta' = S\gamma\theta\beta - \tfrac{1}{2}(S\beta\theta\beta - S\gamma\theta\gamma) \sin 2u$$

or

$$2S\gamma'\theta\beta' = S\nabla\alpha - (S\beta\theta\beta - S\gamma\theta\gamma) \sin 2u$$

as the value of the curvature of the β' curve.

It is easy to see if we set $\beta' = \beta + \gamma$, $\gamma' = \beta - \gamma$, that when

$$S\beta\theta\beta - S\gamma\theta\gamma = 0$$

we have

$$S\beta'\theta\gamma' + S\gamma'\theta\beta' = 0 ,$$

and when

$$S\beta\theta\gamma + S\gamma\theta\beta = 0 , \qquad S\beta'\theta\beta' - S\gamma'\theta\gamma' = 0 .$$

Hence the two sets of extremal directions bisect each other.

20. Recurring to the form $\theta = \alpha\phi_a - \alpha S\zeta$ we need to notice that this form alone with ζ purely arbitrary will not give us θ, since it is too general. The derivation of θ shows that a similar form must hold for any unit vector η, instead of α, if η is fixed with reference to the trihedral of α, β, γ. Hence ζ is not purely arbitrary. In fact, we have from the form given in § 12 for ψ_θ, when θ is taken in the form just above,

$$\theta V\nabla = -V\alpha_2\alpha_1 SV\nabla_1\nabla_2 + \alpha_1 S\nabla_1\zeta(\,) + \alpha SV\nabla\zeta(\,) = -\psi_\theta',$$

whence

$$\psi_\theta = 2\psi_a - V\zeta\phi_a' - V\nabla\zeta S\alpha .$$

But forming $\psi_\theta V\lambda\mu = V\theta'\lambda\theta'\mu$ directly we get

$$\psi_\theta = \psi_a - V\zeta\phi_a'.$$

It follows that we ought to have $V\nabla\zeta = -\psi_a\alpha$. This we can prove directly as follows:

$$V\nabla\zeta = -V\nabla\phi_\beta'\gamma = V\nabla\nabla_2 S\beta_2\gamma = V\nabla_1\nabla_2 S\beta_2\gamma_1 = V\nabla_1\phi_\beta'\gamma_1.$$

Now we have from § 10, $\phi_\beta' = -\phi_a' V\gamma + \phi_\gamma' V\alpha$, and substituting we have:

$$V\nabla\zeta = V\nabla_1[\phi_a' V\gamma_1 V\alpha\beta - \phi_\gamma' V\gamma_1 V\beta\gamma]$$

$$= V\nabla_1[\phi_a'\beta S\alpha\gamma_1 - \phi_a'\alpha S\beta\gamma_1 - \phi_\gamma'\gamma S\beta\gamma_1 + \phi_\gamma'\beta S\gamma\gamma_1]$$

$$= -V\nabla_1\phi_a'\beta S\alpha_1\gamma = V\phi_a'\gamma\phi_a'\beta = -\psi_a\alpha.$$

The form above gives simple forms for the invariants of θ. We have

$$m_1 = p = S\alpha\nabla\alpha - S\alpha\zeta, \qquad m_2 = S\alpha\nabla\zeta + S\zeta\nabla\alpha, \qquad m_3 = S\zeta\nabla\zeta,$$

$$2\epsilon = \alpha S\nabla\alpha + \phi_a\alpha + V\alpha\zeta, \qquad 2\theta\epsilon = \phi_a V\nabla\alpha - \phi_a'\zeta - \zeta S\nabla\alpha.$$

The discussion above can now be stated in terms of α and ζ. For instance: θ cannot have a zero root unless $m_3 = 0$, that is $S\zeta\nabla\zeta = 0$, and the ζ congruence is normal to a set of surfaces. We need not elaborate these results further however.

A SET OF PROPERTIES CHARACTERISTIC OF A CLASS OF CONGRUENCES CONNECTED WITH THE THEORY OF FUNCTIONS*

BY

E. J. WILCZYNSKI

Introduction

In a recent paper,† the author has discussed a class of congruences defined as follows. Let

$$u + iv = w = F(z) = F(x + iy)$$

be a functional relation between the complex variables z and w. Let us first plot corresponding values of z and w in one and the same plane, and let us then project these points upon the unit sphere by stereographic projection. The lines, which join all pairs of points thus obtained upon the sphere, for a given relation $w = F(z)$, form the congruence in question.

If w is not a linear function of z, every congruence of this class has the following properties.

I*a*. *It is a W-congruence whose focal sheets are distinct, non-degenerate, and non-ruled surfaces.*

I*b*. *The focal surfaces are real and have a positive measure of curvature.*

II. *The developables of the congruence determine isothermally conjugate systems of curves on both sheets of the focal surface.*

III. *The asymptotic curves of both sheets of the focal surface belong to linear complexes.*

IV. *The directrix of the first kind, for every point of either focal sheet, coincides with the directrix of the second kind for the corresponding point of the other focal sheet.*

V*a*. *The directrix quadrics of both focal sheets are non-degenerate, and coincide with each other.*

V*b*. *Both of these directrix quadrics coincide with the Riemann sphere.*

If we prefer, we may replace property II by

* Presented to the Society, April 10, 1920.

† E. J. Wilczynski, *Line-geometric representations for functions of a complex variable*, these Transactions, vol. 20 (1919), pp. 271–298. Hereafter quoted as *Line geometric representations*.

II′. *The congruences obtained from the given one by Laplace transformations are also W congruences.**

In the present paper, we propose to show that *these properties are characteristic* of the class of congruences defined by a functional relation between two complex variables on the same sphere;† and, in this connection, it is of interest to observe that properties I*a*, II, III, IV, and V*a*, are *purely projective*, that I*b* is concerned merely with questions of reality, and that *the only metric property in the list is Vb*.

The case of a linear relation between z and w requires a separate discussion, which will be given in Articles 9 and 10.

1. The differential equations of a congruence which possesses properties I*a* and II

Any congruence, whose focal sheets do not coincide, may be studied by means of a completely integrable system of partial differential equations, of the form

$$\begin{gathered} y_v = mz, \qquad z_u = ny, \\ y_{uu} = ay + bz + cy_u + dz_v, \\ z_{vv} = a'y + b'z + c'y_u + d'z_v, \end{gathered} \tag{1}$$

where the coefficients $m, n, a, b, c, d, a', b', c', d'$ are functions of u and v which satisfy the integrability conditions

$$\begin{gathered} c = f_u, \qquad d' = f_v, \qquad b = -d_v - df_v, \qquad a' = -c'_u - c'f_u, \\ W = mn - c'd = f_{uv}, \\ m_{uu} + d_{vv} + df_{vv} + d_vf_v - f_um_u = ma + db', \\ n_{vv} + c'_{uu} + c'f_{uu} + c'_uf_u - f_vn_v = c'a + nb', \\ 2m_un + mn_u = a_v + f_umn + a'd, \\ m_vn + 2mn_v = b'_u + f_vmn + bc',‡ \end{gathered} \tag{2}$$

where f is an arbitrary function of u and v.

* This is a consequence of a theorem first proved by Demoulin and Tzitzéica. See my paper *The general theory of congruences*, these Transactions, vol. 16 (1915), p. 322. This paper will be quoted hereafter as *Congruences*.

† We may adopt this simplified form of statement, even if we *have* made use of a special method of transferring the variables z and w from the plane to the sphere. For, whenever there exists a functional relation between two complex variables on the sphere, corresponding points may be projected stereographically upon the plane, and the original construction may then be applied as indicated.

‡ E. J. Wilczynski, *Sur la théorie générale des congruences*. Mémoire conronné par la classe des sciences. Mémoires publiés par la Classe des Sciences de l'Académie Royale de Belgique. Collection en 4°. Deuxième série. Tome III (1911). This paper will hereafter be cited as the *Brussels Paper*.

The reason for this is simple. Under conditions (2), system (1) will have exactly four linearly independent solutions $(y^{(k)}, z^{(k)})$, $(k = 1, 2, 3, 4)$, such that the general solution will be of the form

$$y = \sum_{k=1}^{4} c^{(k)} y^{(k)}, \qquad z = \sum_{k=1}^{4} c^{(k)} z^{(k)},$$

where $c^{(1)}, \cdots, c^{(4)}$ are constants. Let $y^{(1)}, \cdots, y^{(4)}$ and $z^{(1)}, \cdots, z^{(4)}$ be interpreted as the homogeneous coördinates of two points, P_y and P_z. As u and v vary, P_y and P_z will describe two surfaces, S_y and S_z (either or both of which may degenerate into curves), and the line $P_y P_z$ will generate a congruence. The surfaces S_y and S_z will be the focal surfaces of the congruence, and the ruled surfaces obtained by equating either u or v to a constant will be its developables.*

The *invariants* and *covariants* of (1) are those functions of the coefficients and variables which are left unchanged, absolutely or except for a factor, when system (1) is subjected to any transformation of the form

$$(3) \qquad y = \lambda(u)\bar{y}, \qquad z = \mu(v)\bar{z}, \qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v),$$

where λ, μ, α, β are arbitrary functions of the single variables indicated. The effect of this transformation upon the coefficients of (1) is given by the following equations:†

$$\overline{m} = \frac{\mu}{\lambda\beta_v} m, \qquad \bar{n} = \frac{\lambda}{\mu\alpha_u} n,$$

$$(4) \qquad \begin{aligned} \bar{a} &= \frac{1}{\alpha_u^2}\left(a + \frac{\lambda_u}{\lambda} c - \frac{\lambda_{uu}}{\lambda}\right), & \bar{a}' &= \frac{\lambda}{\mu\beta_v^2}\left(a' + \frac{\lambda_u}{\lambda} c'\right), \\ \bar{b} &= \frac{\mu}{\lambda\alpha_u^2}\left(b + \frac{\mu_v}{\mu} d\right), & \bar{b}' &= \frac{1}{\beta_v^2}\left(b' + \frac{\mu_v}{\mu} d' - \frac{\mu_{vv}}{\mu}\right), \\ \bar{c} &= \frac{1}{\alpha_u}\left(c - 2\frac{\lambda_u}{\lambda} - \frac{\alpha_{uu}}{\alpha_u}\right), & \bar{c}' &= \frac{\lambda\alpha_u}{\mu\beta_v^2} c', \\ \bar{d} &= \frac{\mu\beta_v}{\lambda\alpha_u^2} d, & \bar{d}' &= \frac{1}{\beta_v}\left(d' - 2\frac{\mu_v}{\mu} - \frac{\beta_{vv}}{\beta_v}\right). \end{aligned}$$

A W-congruence is one which makes the asymptotic curves upon the two sheets of the focal surface correspond, and is characterized by the condition‡

$$(5) \qquad W = mn - c'd = f_{uv} = 0.$$

Thus, if our congruence is a W-congruence, we may put

$$f = U + V,$$

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* *Brussels Paper*, pp. 16 and 17.

† Obtained by combining (16) and (22) of the *Brussels Paper*.

‡ *Brussels Paper*, p. 46.

where U and V are functions of u and v *alone*, respectively. We then have

$$c = f_u = U', \qquad d' = f_v = V'.$$

As (4) shows we can find infinitely many transformations of form (3) for which $\bar{c}$ and $\bar{d}'$ will both be equal to zero. We shall assume that such a transformation has been made. We shall then have

$$c = d' = 0, \qquad mn - c'd = 0, \tag{6}$$

and $f = U + V$ will be a *constant*. Equations (4) show that there are infinitely many transformations of form (3) which will preserve these conditions, namely all those for which

$$2\frac{\lambda_u}{\lambda} + \frac{\alpha_{uu}}{\alpha_u} = 0, \qquad 2\frac{\mu_v}{\mu} + \frac{\beta_{vv}}{\beta_v} = 0. \tag{7}$$

The developables of the congruence, $u =$ const. and $v =$ const., always determine a conjugate system upon the focal surfaces. The conjugate system thus determined on S_y is *isothermally* conjugate if and only if

$$\frac{\partial^2 \log d/m}{\partial u \partial v} = 0,^*$$

that is, if and only if d/m is of the form

$$\frac{d}{m} = U(u)V(v),$$

where U and V are non-vanishing functions of the *single* variables indicated.† But if this is so, let us make a transformation of form (3), conditioned by (7) so as not to disturb the simplification (6) already established. We find, from (4),

$$\frac{\bar{d}}{\bar{m}} = \frac{\beta_v^2}{\alpha_u^2} UV.$$

If we put $\alpha_u^2 = U, \beta_v^2 V = 1$, we shall have $\bar{d} = \bar{m}$, and then from (6) $\bar{c}' = \bar{n}$. That is, the developables of the congruence will determine an *isothermally* conjugate system on S_z as well. Let us assume such a transformation made, so that we have

$$c = d' = 0, \qquad d = m, \qquad c' = n. \tag{8}$$

The most general transformation of form (3) which preserves these relations will be conditioned by (7) and

$$\alpha_u^2 = \beta_v^2.$$

* *Congruences*, p. 322.

† Non-vanishing because the focal sheets are assumed to be non-developable and non-degenerate.

But α_u is a function of u alone, and β_v is a function of v alone. Therefore this last condition will involve a contradiction unless the common value of α_u^2 and β_v^2 is a constant. On account of (3) this constant must be different from zero, and (7) shows that $\lambda(u)$ and $\mu(v)$ must then also be non-vanishing constants. Thus we find that the most general transformation of form (3), which preserves the conditions (8), is conditioned by

$$\lambda = c_1, \qquad \mu = c_2, \qquad \alpha_u = \pm \beta_v = c_3, \tag{9}$$

where c_1, c_2, c_3 are arbitrary, non-vanishing, constants.

Making use of (2) and (8) we find the relations

$$\begin{gathered} c = 0, \qquad d' = 0, \qquad c' = n, \qquad d = m, \\ a' = -n_u, \qquad b = -m_v, \\ m_{uu} + m_{vv} = m(a + b'), \qquad n_{uu} + n_{vv} = n(a + b'), \\ 2(m_u n + m n_u) = a_v, \qquad 2(m_v n + m n_v) = b'_u, \end{gathered} \tag{10}$$

between the coefficients of a system of form (1) which has the required properties.

We can simplify these conditions considerably. The last two equations of (10) imply the existence of two functions p and q, of u and v, such that

$$\begin{gathered} 2mn = p_v, \qquad a = p_u, \\ 2mn = q_u, \qquad b' = p_v; \end{gathered}$$

but this implies further $p_v = q_u$, which requires the existence of a function $r(u, v)$ such that

$$p = r_u, \qquad q = r_v, \qquad a = r_{uu}, \qquad b' = r_{vv}.$$

Thus we have found the following result. *The differential equations of a W-congruence, whose focal sheets are distinct non-degenerate and non-developable surfaces, and whose developables determine an isothermally conjugate system of curves upon both sheets of the focal surface can be written in the form* (1) *with the coefficients*

$$\begin{aligned} a &= r_{uu}, & b &= -m_v, & c &= 0, & d &= m \neq 0,^* \\ a' &= -n_u, & b' &= r_{vv}, & c' &= n \neq 0, & d' &= 0, \end{aligned} \tag{11}$$

where m, n, and r are functions of u and v, which satisfy the relations

$$\begin{gathered} m_{uu} + m_{vv} = m(r_{uu} + r_{vv}), \qquad n_{uu} + n_{vv} = n(r_{uu} + r_{vv}), \\ 2mn = r_{uv}. \end{gathered} \tag{12}$$

* The conditions $m \neq 0$, $n \neq 0$, $c' \neq 0$, $d \neq 0$ insure the non-degenerate and non-developable character of the focal sheets.

Moreover, the most general transformation of form (3), *which will preserve this form of system* (1), *is conditioned by*

$$\lambda = c_1, \qquad \mu = c_2, \qquad \alpha_u = \pm \beta_v = c_3, \tag{9}$$

where c_1, c_2, c_3 are arbitrary, non-vanishing, constants.

2. Differential equations of the focal sheets referred to their asymptotic lines

The differential equations of S_y are found from (1) by eliminating the function z and its partial derivatives. If we make use of (11), we find these equations to be

$$y_{uu} - y_{vv} = r_{uu}\, y - 2\frac{m_v}{m} y_v, \qquad y_{uv} = \tfrac{1}{2} r_{uv}\, y + \frac{m_u}{m} y_v. \tag{13}$$

To find the asymptotic lines on S_y we must find two independent solutions of

$$m\theta_u^2 + d\theta_v^2 = 0^* \tag{14a}$$

and equate them to arbitrary constants. In our case $d = m \neq 0$, so that if we put

$$\bar{u} = u + iv, \qquad \bar{v} = u - iv, \tag{14b}$$

the asymptotic lines of S_y will be given by $\bar{u} =$ const. and $\bar{v} =$ const. If we introduce $\bar{u}$ and $\bar{v}$ as independent variables into (13), we find *the differential equations*

$$\begin{aligned} y_{\bar{u}\bar{u}} + 2a_1 y_{\bar{u}} + 2b_1 y_{\bar{v}} + c_1 y &= 0, \\ y_{\bar{v}\bar{v}} + 2a_1' y_{\bar{u}} + 2b_1' y_{\bar{v}} + c_1' y &= 0, \end{aligned} \tag{15}$$

of S_y referred to its asymptotic lines, where

$$\begin{aligned} a_1 &= -\frac{1}{2}\frac{m_{\bar{v}}}{m}, \qquad b_1 = +\frac{1}{2}\frac{m_{\bar{u}}}{m}, \\ c_1 &= -\tfrac{1}{4}(a - 2imn) = -\tfrac{1}{2}(r_{\bar{u}\bar{u}} + r_{\bar{u}\bar{v}}), \\ a_1' &= +\frac{1}{2}\frac{m_{\bar{v}}}{m}, \qquad b_1' = -\frac{1}{2}\frac{m_{\bar{v}}}{m}, \\ c_1' &= -\tfrac{1}{4}(a + 2imn) = -\tfrac{1}{2}(r_{\bar{u}\bar{v}} + r_{\bar{v}\bar{v}}), \end{aligned} \tag{16}$$

and where we have used the relations (12) *in their new form*

$$m_{\bar{u}\bar{v}} = mr_{\bar{u}\bar{v}}, \qquad n_{\bar{u}\bar{v}} = nr_{\bar{u}\bar{v}}, \qquad 2mn = i(r_{\bar{u}\bar{u}} - r_{\bar{v}\bar{v}}). \tag{17}$$

In the same way we find the equations of S_z to be

$$-z_{uu} + z_{vv} = r_{vv}\, z - 2\frac{n_u}{n} z_u, \qquad z_{uv} = \frac{1}{2} r_{uv}\, z + \frac{n_v}{n} z_u. \tag{18}$$

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* *Brussels Paper*, p. 46.

Since the congruence is a W-congruence, the variables $\bar{u}$ and $\bar{v}$ will determine the asymptotic lines on S_z as well as on S_y. If we introduce these variables we obtain a system of differential equations for S_z of the same form as (15) but with the coefficients:

$$(19)\qquad \begin{aligned} a_2 &= -\frac{1}{2}\frac{n_{\bar{u}}}{n}, & b_2 &= -\frac{1}{2}\frac{n_{\bar{u}}}{n}, & c_2 &= -\frac{1}{2}(r_{\bar{u}\bar{u}} - r_{\bar{u}\bar{v}}), \\ a_2' &= -\frac{1}{2}\frac{n_{\bar{v}}}{n}, & b_2' &= -\frac{1}{2}\frac{n_{\bar{v}}}{n}, & c_2' &= +\frac{1}{2}(r_{\bar{u}\bar{v}} - r_{\bar{v}\bar{v}}). \end{aligned}$$

3. Introduction of property III

If S_y is not a ruled surface, a_1' and b_1 will be different from zero, and consequently the same thing must be true of $m_{\bar{u}}$ and $m_{\bar{v}}$. The asymptotic lines of such a surface will belong to linear complexes, if and only if

$$(20)\qquad \frac{\partial^2 \log a_1'}{\partial\bar{u}\partial\bar{v}} = \frac{\partial^2 \log b_1}{\partial\bar{u}\partial\bar{v}} = 4a_1' b_1,\text{*}$$

that is, if and only if

$$(21)\qquad \begin{aligned} \frac{m_{\bar{u}\bar{u}\bar{v}}}{m_{\bar{u}}} - \frac{m_{\bar{u}\bar{v}}\, m_{\bar{u}\bar{u}}}{m_{\bar{u}}^2} - \frac{m_{\bar{u}\bar{v}}}{m} &= 0, \\ \frac{m_{\bar{u}\bar{v}\bar{v}}}{m_{\bar{v}}} - \frac{m_{\bar{u}\bar{v}}\, m_{\bar{v}\bar{v}}}{m_{\bar{v}}^2} - \frac{m_{\bar{u}\bar{v}}}{m} &= 0. \end{aligned}$$

These conditions are satisfied in the first place if

$$(22)\qquad m_{\bar{u}\bar{v}} = 0, \qquad m_{\bar{u}} \neq 0, \qquad m_{\bar{v}} \neq 0,$$

that is, if

$$(23)\qquad m = U_1 + V_1,$$

where U_1 and V_1 denote arbitrary functions of the single variables $\bar{u}$ and $\bar{v}$ respectively.

If $m_{\bar{u}\bar{v}} \neq 0$, we may write (21) as follows:

$$(24)\qquad \begin{aligned} \frac{m_{\bar{u}\bar{u}\bar{v}}}{m_{\bar{u}\bar{v}}} - \frac{m_{\bar{u}\bar{u}}}{m_{\bar{u}}} - \frac{m_{\bar{u}}}{m} &= 0, \\ \frac{m_{\bar{u}\bar{v}\bar{v}}}{m_{\bar{u}\bar{v}}} - \frac{m_{\bar{v}\bar{v}}}{m_{\bar{v}}} - \frac{m_{\bar{v}}}{m} &= 0, \end{aligned}$$

whence

$$(25)\qquad \frac{m_{\bar{u}\bar{v}}}{m m_{\bar{u}}} = V_1', \qquad \frac{m_{\bar{u}\bar{v}}}{m m_{\bar{v}}} = U_1',$$

where U_1' and V_1' are arbitrary functions of the single variables $\bar{u}$ and $\bar{v}$ re-

* C. T. Sullivan, *Properties of surfaces whose asymptotic curves belong to linear complexes*, these Transactions, vol. 15 (1914), p. 175.

spectively, and where the notation indicates that we shall consider presently the functions U_1 and V_1, whose derivatives are U_1' and V_1' respectively. From (25) we see that

$$V_1' m_{\bar{u}} = U_1' m_{\bar{v}},$$

and from this partial differential equation we conclude that m must be a function of $U_1 + V_1$ alone. Thus we have

$$(26) \qquad m = f(w_1), \qquad w_1 = U_1 + V_1.$$

But from (26) we find

$$(27) \qquad m_{\bar{u}} = f'(w_1) U_1', \qquad m_{\bar{v}} = f'(w_1) V_1', \qquad m_{\bar{u}\bar{v}} = f''(w_1) U_1' V_1'.$$

If we substitute these values into (25), we find that $f(w_1)$ must satisfy the differential equation

$$(28) \qquad f'' = ff',$$

where we may assume $f' \neq 0, f'' \neq 0$, since we are considering the case $m_{\bar{u}\bar{v}} \neq 0$. This differential equation is easy to integrate. We find first

$$(29) \qquad 2f' = f^2 - a^2,$$

where a is an arbitrary constant, and then

$$(30) \qquad f(w_1) = \frac{-2}{w_1 - c} \quad \text{if} \quad a = 0,$$

and

$$(31) \qquad f(w_1) = a\frac{1 + e^{a(w_1 - c)}}{1 - e^{a(w_1 - c)}} \quad \text{if} \quad a \neq 0,$$

where c, in both cases, represents a new arbitrary constant.

In this discussion we have assumed that S_y is not a ruled surface. If S_y is a ruled surface, at least one of the quantities $m_{\bar{u}}$ and $m_{\bar{v}}$ is equal to zero. If $m_{\bar{u}} = 0$, we have $b_1 = 0$, and the curves $\bar{v} =$ const. are the generators of S_y. In that case the condition $m_{\bar{u}} = 0$ replaces the first of (21). The second condition (21) does not lose its significance unless $m_{\bar{v}}$ is equal to zero also, but it is satisfied by $m_{\bar{u}} = 0$ and may therefore be omitted. Similarly, if $m_{\bar{v}} = 0$, $m_{\bar{u}} \neq 0$, (21) is replaced by $m_{\bar{v}} = 0$. If $m_{\bar{u}} = m_{\bar{v}} = 0$, S_y is a quadric, and these two conditions replace (21). Since, however, in all of these cases we have $m_{\bar{u}\bar{v}} = 0$, we may think of them as included in our original discussion.

If the asymptotic lines of S_z also belong to linear complexes, we find again three possibilities. We shall have either

$$(32) \qquad n = U_2 + V_2 = w_2,$$

where U_2 and V_2 are functions of $\bar{u}$ and $\bar{v}$ alone respectively, or

$$n = g(U_2 + V_2) = g(w_2), \tag{33}$$

where

$$2g' = g^2 - b^2 \tag{34}$$

if b is an arbitrary constant, so that

$$g(w_2) = \frac{-2}{w_1 - d}, \quad \text{if} \quad b = 0, \tag{35}$$

or

$$g(w_2) = b\,\frac{1 + e^{b(w_2-d)}}{1 - e^{b(w_2-d)}}, \quad \text{if} \quad b \neq 0. \tag{36}$$

In order to decide whether S_y and S_z both may have this property, it remains to investigate whether it is possible to satisfy the integrability conditions (17) with such expressions for m and n. But we shall postpone this investigation until we have found out the geometric significance of the distinction between the cases $m_{\bar{u}\bar{v}} = 0$ and $m_{\bar{u}\bar{v}} \neq 0$.

4. Determination of the directrices of the focal sheets. Property IV

The osculating linear complexes of the two asymptotic curves which meet at any point of a *non-ruled surface* are uniquely determined and have in common a linear congruence. One of the directrices of this congruence lies in the tangent plane of the surface point under consideration; the other passes through the point itself.* These two lines are called the *directrices*, of the first and second kind respectively, of the surface point. We propose to find the equations of these directrices for both of the focal sheets of a congruence of the kind under discussion. We shall then be in a position to impose the further restriction upon the congruence which is expressed by property IV.

The equations of these directrices may be taken over from the projective theory of surfaces, referred to a local tetrahedron of reference determined by the surface considered. Since we are studying two surfaces, S_y and S_z, and since the two local coördinate systems will be quite distinct, it then remains to determine the relation between the two coördinate systems. For the sake of symmetry and simplicity we shall introduce a third coördinate system, the local coördinate system of the congruence, and refer finally the equations of all four directrices to this system.

The local coördinate system of the congruence has y and z as two of its fundamental points. The other two vertices of its tetrahedron of reference

* E. J. Wilczynski, *Projective differential geometry of curved surfaces* (Second Memoir), these Transactions, vol. 9 (1908), p. 95. This paper will be quoted as *Second Memoir* hereafter. The *First Memoir* is in these Transactions, vol. 8 (1907).

are given by

$$\rho = y_u - \frac{m_u}{m} y, \qquad \sigma = z_v - \frac{n_v}{n} z. \tag{37}$$

The points P_ρ and P_σ are obtained from y and z by Laplace's transformation. The local coördinates (x_1, x_2, x_3, x_4) of a point are then determined, except for a common factor, by the expression

$$t = x_1 y + x_2 z + x_3 \rho + x_4 \sigma \tag{38}$$

which represents such a point. To make this a little clearer, let us remember that system (1) has four linearly independent solutions $(y^{(k)}, z^{(k)})$, $(k = 1, 2, 3, 4)$. By substituting these solutions in (37) we find four functions $\rho^{(k)}$ and four functions $\sigma^{(k)}$, which determine two new points, P_ρ and P_σ. Consequently any expression of form (38) also determines a point P_t. It is the point determined in this way whose local coördinates are $(x_1, \cdots, x_4)$.

The local tetrahedron of the surface S_y is determined by the four expressions

$$\begin{gathered} y_1 = y, \qquad z_1 = y_{\bar u} + a_1 y, \qquad \rho_1 = y_v + b_1' y, \\ \sigma_1 = y_{\bar u \bar v} + b_1' y_{\bar u} + a_1 y_{\bar v} + \tfrac{1}{2}[(a_1)_{\bar v} + (b_1')_{\bar u} + 2a_1 b_1'] y, \end{gathered} \tag{39}$$ *

while that of S_z is given by

$$\begin{gathered} y_2 = z, \qquad z_2 = z_{\bar u} + a_2 z, \qquad \rho_2 = z_{\bar v} + b_2' z, \\ \sigma_2 = z_{\bar u \bar v} + b_2' z_{\bar u} + a_2 z_{\bar v} + \tfrac{1}{2}[(a_2)_{\bar v} + (b_2')_{\bar u} + 2a_2 b_2'] z. \end{gathered} \tag{40}$$

We proceed to express these quantities in terms of y, z, ρ, and σ. We find

$$\begin{aligned} y_1 &= y, \\ z_1 &= \frac{1}{4}\left(\frac{m_u}{m} + i\,\frac{m_v}{m}\right) y - \frac{1}{2}\, imz + \frac{1}{2}\rho, \\ \rho_1 &= \frac{1}{4}\left(\frac{m_u}{m} - i\,\frac{m_v}{m}\right) y + \frac{1}{2}\, imz + \frac{1}{2}\rho, \\ \sigma_1 &= \frac{1}{4}\left(r_{uu} - \frac{m_u^2}{m^2} + \frac{m_{\bar u}\, m_{\bar v}}{m^2} - 2\,\frac{\partial^2 \log m}{\partial \bar u \partial \bar v}\right) y \\ &\qquad + \frac{1}{4}\left(2m\,\frac{n_v}{n} - m_v\right) z - \frac{1}{4}\,\frac{m_u}{m}\rho + \frac{1}{2} m\sigma, \end{aligned} \tag{41}$$

* *First Memoir*, p. 248.

and

$$
\begin{aligned}
y_2 &= z, \\
z_2 &= \frac{1}{2}ny - \frac{1}{4}\left(\frac{n_u}{n} + i\frac{n_v}{n}\right)z - \frac{1}{2}i\sigma, \\
\rho_2 &= \frac{1}{2}ny - \frac{1}{4}\left(\frac{n_u}{n} - i\frac{n_v}{n}\right)z + \frac{1}{2}i\sigma, \\
\sigma_2 &= \frac{1}{4}\left(2n\frac{m_u}{m} - n_u\right)y + \frac{1}{4}\left(r_{vv} - \frac{n_v^2}{n^2} + \frac{n_{\bar u}n_{\bar v}}{n^2} - 2\frac{\partial^2 \log n}{\partial\bar u\partial\bar v}\right)z \\
&\qquad + \frac{1}{2}n\rho - \frac{1}{4}\frac{n_v}{n}\sigma.
\end{aligned} \tag{42}
$$

Let the coördinates of the point (x_1, x_2, x_3, x_4) when referred to the local system of the surface S_y be called $(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)})$. Then we must have identically

$$\omega(x_1 y + x_2 z + x_3\rho + x_4\sigma) = x_1^{(1)}y_1 + x_2^{(1)}z_1 + x_3^{(1)}\rho_1 + x_4^{(1)}\sigma_1.$$

If we introduce the expressions (41) in the right member of this equation, and then equate the coefficients of y, z, ρ, σ in the two members, we obtain the following equations of transformation,

$$
\begin{aligned}
\omega x_1 &= x_1^{(1)} + \frac{1}{4}\left(\frac{m_u}{m} + i\frac{m_v}{m}\right)x_2^{(1)} + \frac{1}{4}\left(\frac{m_u}{m} - i\frac{m_v}{m}\right)x_3^{(1)} \\
&\qquad + \frac{1}{4}\left(r_{uu} - \frac{m_u^2}{m^2} + \frac{m_{\bar u}m_{\bar v}}{m^2} - 2\frac{\partial^2\log m}{\partial\bar u\partial\bar v}\right)x_4^{(1)}, \\
\omega x_2 &= -\frac{1}{2}imx_2^{(1)} + \frac{1}{2}imx_3^{(1)} + \frac{1}{4}\left(2m\frac{n_v}{n} - m_v\right)x_4^{(1)}, \\
\omega x_3 &= \frac{1}{2}x_2^{(1)} + \frac{1}{2}x_3^{(1)} - \frac{1}{4}\frac{m_u}{m}x_4^{(1)}, \\
\omega x_4 &= \tfrac{1}{2}mx_4^{(1)},
\end{aligned} \tag{43}
$$

where ω, the factor of proportionality, may be chosen arbitrarily. We make use of this fact in writing the inverse transformation as follows:

$$
\begin{aligned}
x_1^{(1)} &= 4mx_1 + 2\frac{m_v}{m}x_2 - 2m_u x_3 \\
&\qquad + \left(\frac{m_{uu} + m_{vv}}{m} - \frac{m_u^2 + m_v^2}{2m^2} - 2\frac{m_v n_v}{mn} - 2r_{uu}\right)x_4, \\
x_2^{(1)} &= 4ix_2 + 4mx_3 + 2\left(\frac{m_u}{m} + i\frac{m_v}{m} - 2i\frac{n_v}{n}\right)x_4, \\
x_3^{(1)} &= -4ix_2 + 4mx_3 + 2\left(\frac{m_u}{m} - i\frac{m_v}{m} + 2i\frac{n_v}{n}\right)x_4, \\
x_4^{(1)} &= 8x_4.
\end{aligned} \tag{44}
$$

In precisely the same way we denote by $(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_4^{(2)})$ the coördinates of the point (x_1, x_2, x_3, x_4), when referred to the local system of the surface S_z. We find

$$\omega' x_1 = \frac{1}{2} n x_2^{(2)} + \frac{1}{2} n x_3^{(2)} + \frac{1}{4}\left(2n\frac{m_u}{m} - n_u\right) x_4^{(2)},$$

$$\omega' x_2 = x_1^{(2)} - \frac{1}{4}\left(\frac{n_u}{n} + i\frac{n_v}{n}\right) x_2^{(2)} - \frac{1}{4}\left(\frac{n_u}{n} - i\frac{n_v}{n}\right) x_3^{(2)}$$

$$(45) \qquad + \frac{1}{4}\left(r_{vv} - \frac{n_v^2}{n^2} + \frac{n_{\bar u} n_{\bar v}}{n^2} - 2\frac{\partial^2 \log n}{\partial \bar u \partial \bar v}\right) x_4^{(2)},$$

$$\omega' x_3 = \tfrac{1}{2} n x_4^{(2)},$$

$$\omega' x_4 = -\frac{i}{2} x_2^{(2)} + \frac{i}{2} x_3^{(2)} - \frac{1}{4}\frac{n_v}{n} x_4^{(2)},$$

and

$$x_1^{(2)} = 2\frac{n_u}{n} x_1 + 4n x_2$$

$$+ \left(\frac{n_{uu} + n_{vv}}{n} - \frac{n_u^2 + n_v^2}{2n^2} - 2\frac{m_u n_u}{mn} - 2r_{vv}\right) x_3 - 2n_v x_4,$$

$$(46) \qquad x_2^{(2)} = 4x_1 + 2\left(\frac{n_u}{n} + i\frac{n_v}{n} - 2\frac{m_u}{m}\right) x_3 + 4inx_4,$$

$$x_3^{(2)} = 4x_1 + 2\left(\frac{n_u}{n} - i\frac{n_v}{n} - 2\frac{m_u}{m}\right) x_3 - 4inx_4,$$

$$x_4^{(2)} = 8x_3.$$

The directrix d_1 of the first kind of P_y has the equations

$$(47) \qquad x_4^{(1)} = 0, \qquad 2x_1^{(1)} + \frac{(a_1')_{\bar u}}{a_1'} x_2^{(1)} + \frac{(b_1)_{\bar v}}{b_1} x_3^{(1)} = 0,$$

and the equations of d_1', the directrix of the second kind of P_y, are

$$(48) \qquad 2x_2^{(1)} + \frac{(b_1)_{\bar v}}{b_1} x_4^{(1)} = 0, \qquad 2x_3^{(1)} + \frac{(a_1')_{\bar u}}{a_1'} x_4^{(1)} = 0,^*$$

both referred to the local coördinate system of P_y. Moreover we have, in our case,

$$(49) \qquad \frac{(a_1')_{\bar u}}{a_1'} = \frac{m_{\bar u \bar v}}{m_{\bar v}} - \frac{m_{\bar u}}{m}, \qquad \frac{(b_1)_{\bar v}}{b_1} = \frac{m_{\bar u \bar v}}{m_{\bar u}} - \frac{m_{\bar v}}{m}.$$

Thus the equations of d_1 referred to the local coördinate system of the congruence reduce to

$$(50) \qquad x_4 = 0, \qquad 2x_1 + \frac{m_v}{m}\frac{m_{\bar u \bar v}}{m_{\bar u} m_{\bar v}} x_2 + m_u\left(\frac{m_{\bar u \bar v}}{m_{\bar u} m_{\bar v}} - \frac{2}{m}\right) x_3 = 0,$$

* *Second Memoir*, p. 95.

and those of d_1' become

$$(51) \qquad 2x_2 + \left(m_v \frac{m_{\bar{u}\bar{v}}}{m_{\bar{u}}\, m_{\bar{v}}} - 2\frac{n_v}{n} \right) x_4 = 0, \qquad 2x_3 + \frac{m_u}{m}\frac{m_{\bar{u}\bar{v}}}{m_{\bar{u}}\, m_{\bar{v}}} x_4 = 0.$$

Let d_2 and d_2' be the directrices of the first and second kind respectively of P_z. The equations of d_2 are

$$(52) \qquad x_3 = 0, \qquad \frac{n_u}{n}\frac{n_{\bar{u}\bar{v}}}{n_{\bar{u}}\, n_{\bar{v}}} x_1 + 2x_2 + n_v \left(\frac{n_{\bar{u}\bar{v}}}{n_{\bar{u}}\, n_{\bar{v}}} - \frac{2}{n} \right) x_4 = 0,$$

and those of d_2',

$$(53) \qquad 2x_1 + \left(n_u \frac{n_{\bar{u}\bar{v}}}{n_{\bar{u}}\, n_{\bar{v}}} - 2\frac{m_u}{m} \right) x_3 = 0, \qquad 2x_4 + \frac{n_v}{n}\frac{n_{\bar{u}\bar{v}}}{n_{\bar{u}}\, n_{\bar{v}}} x_3 = 0.$$

The line d_1 will contain $P_z(0, 1, 0, 0)$, and d_2 will contain $P_y(1, 0, 0, 0)$ if and only if

$$(54) \qquad m_v\, m_{\bar{u}\bar{v}} = 0, \qquad n_u\, n_{\bar{u}\bar{v}} = 0.$$

If $m_{\bar{u}\bar{v}} = 0$, we see from (17) that $n_{\bar{u}\bar{v}} = 0$ also, and vice versa. Therefore we have to consider the two cases

Case A $\qquad m_{\bar{u}\bar{v}} = 0, \qquad n_{\bar{u}\bar{v}} = 0,$

and

Case B $\qquad m_v = n_u = 0.$

In case A we see that d_1 and d_2' on the one hand, and d_2 and d_1' on the other coincide. Thus, such congruences possess property IV as well as properties I*a*, II, and III. We shall show now that case B can present itself only when the conditions of case A are satisfied at the same time. In case B we have

$$m = U(u), \qquad n = V(v),$$

where U and V are functions of u and v alone, respectively. Therefore the integrability conditions (12) reduce to

$$(55) \qquad U'' = U(r_{uu} + r_{vv}), \qquad V'' = V(r_{uu} + r_{vv}), \qquad 2UV = r_{uv}.$$

If $r_{uu} + r_{vv} = 0$, these conditions give

$$m = U = c_1 u + c_0, \qquad n = V = d_1 v + d_0,$$

$$r_{uv} = 2(c_1 u + c_0)(d_1 v + d_0),$$

where c_1, c_0, d_1, d_0 are arbitrary constants. From the last equation we find

$$r_u = \frac{1}{d_1}(c_1 u + c_0)(d_1 v + d_0)^2 + U_1',$$

$$r_v = \frac{1}{c_1}(c_1 u + c_0)^2 (d_1 v + d_0) + V_1',$$

where U_1' and V_1' are functions of u and v alone respectively, and therefore

$$r_{uu} + r_{vv} = U_1'' + V_1'' + \frac{c_1}{d_1}(d_1 v + d_0)^2 + \frac{d_1}{c_1}(c_1 u + c_0)^2 = 0.$$

This equation leads to a contradiction (the equality of a function of u alone with a function of v alone), unless

$$U_1'' = k - \frac{d_1}{c_1}(c_1 u + c_0)^2, \qquad V_1'' = -k - \frac{c_1}{d_1}(d_1 v + d_0)^2,$$

where k is an arbitrary constant. The coefficients of a system of form (1) corresponding to these conditions have therefore been determined, but we see at once that they also satisfy the conditions $m_{\bar{u}\bar{v}} = n_{\bar{u}\bar{v}} = 0$ of case A.

Let us then assume $r_{uu} + r_{vv} \neq 0$. The first two equations of (55) show us that we must have

$$\frac{U''}{U} = \frac{V''}{V} = r_{uu} + r_{vv}, \tag{56}$$

and this equality can subsist only if the common value of the two fractions is a constant, say k^2. We may moreover assume $k \neq 0$, since we have just discussed the case $U'' = V'' = 0$. But from

$$U'' = k^2 U, \qquad V'' = k^2 V,$$

follows

$$U = c_1 e^{ku} + c_2 e^{-ku}, \qquad V = d_1 e^{kv} + d_2 e^{-kv},$$

where c_1, c_2, d_1, d_2 are arbitrary constants. The third equation of (55) gives

$$r_{uv} = 2(c_1 e^{ku} + c_2 e^{-ku})(d_1 e^{kv} + d_2 e^{-kv}),$$

whence

$$r_u = \frac{2}{k}(c_1 e^{ku} + c_2 e^{-ku})(d_1 e^{kv} - d_2 e^{-kv}) + U_1',$$

$$r_v = \frac{2}{k}(c_1 e^{ku} - c_2 e^{-ku})(d_1 e^{kv} + d_2 e^{-kv}) + V_1',$$

where again U_1' and V_1' are functions of u and v alone respectively. Consequently we find

$$r_{uu} = 2(c_1 e^{ku} - c_2 e^{-ku})(d_1 e^{kv} - d_2 e^{-kv}) + U_1'',$$

$$r_{vv} = 2(c_1 e^{ku} - c_2 e^{-ku})(d_1 e^{kv} - d_2 e^{-kv}) + V_1''.$$

But, according to (56) we must have, in this case,

$$r_{uu} + r_{vv} = k^2;$$

consequently we obtain the condition

$$4(c_1 e^{ku} - c_2 e^{-ku})(d_1 e^{kv} - d_2 e^{-kv}) + U_1'' + V_1'' = k^2,$$

whence, by differentiation,

$$4k(c_1 e^{ku} + c_2 e^{-ku})(d_1 e^{kv} - d_2 e^{-kv}) + U_1''' = 0,$$

$$4k(c_1 e^{ku} - c_2 e^{-ku})(d_1 e^{kv} + d_2 e^{-kv}) + V_1''' = 0.$$

Since S_y is a non-degenerate surface, c_1 and c_2 cannot both be equal to zero. Since, moreover, we are discussing the case $k \neq 0$, the first of these equations would imply that $d_1 e^{kv} - d_2 e^{-kv}$ is equal to a function of u alone, a contradiction unless this function reduces to a constant D. But this implies further, as a result of a differentiation with respect to v, that

$$k(d_1 e^{kv} + d_2 e^{-kv}) = 0,$$

which is impossible unless d_1 and d_2 are both zero, contrary to our assumption that S_z is a non-degenerate surface.

Therefore; *the only congruences which possess properties Ia, II, III, and IV, are those which correspond to the conditions*

$$m_{\bar{u}\bar{v}} = n_{\bar{u}\bar{v}} = 0.$$

5. The reality property I*b*

Let us consider a congruence with real and distinct, non-degenerate and non-developable focal surfaces, and real developables. Let us use a real coördinate system and let $(y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)})$ and $(z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)})$ be the coördinates of corresponding points P_y and P_z of the two focal sheets, S_y and S_z. Moreover let u and v be the variables which, equated to constants, furnish the two families of developables of the congruence. Since these developables are real, it will be possible to choose u and v as *real* variables; since S_y and S_z are real it will be possible to choose $y^{(k)}$ and $z^{(k)}$ as *real* functions of u and v; in fact the ratios of the $y^{(k)}$'s and of the $z^{(k)}$'s would *have* to be real, if we use a real coördinate system.

Since S_y and S_z are the focal surfaces of the congruence, the four pairs of functions $(y^{(k)}, z^{(k)})$ must satisfy two equations of the form

$$\frac{\partial y}{\partial v} + \alpha y = \omega z, \qquad \frac{\partial z}{\partial u} + \beta z = \omega' z$$

with *real* coefficients, and the real transformation

$$\bar{y} = y e^{\int \alpha dv}, \qquad \bar{z} = z e^{\int \beta du}$$

will transform these equations into

$$\frac{\partial \bar{y}}{\partial v} = m\bar{z}, \qquad \frac{\partial \bar{z}}{\partial u} = n\bar{y},$$

where m and n will be *real* functions of u and v. If we replace the original variables, y and z, by $\bar{y}$ and $\bar{z}$, it is clear that all of the coefficients of the system (1) obtained in this way will be real functions of the real variables u and v.

In simplifying this system of differential equations after assuming that it possessed properties Ia and IIa, we made a number of transformations of form (3). One of these transformations consisted in reducing c and d' to zero. Equations (4) show that this may be accomplished, in infinitely many ways, by means of *real* transformations. We then observed that the ratio $d : m$ was of the form $U(u)\,V(v)$ and made a further transformation, determined by

$$\alpha_u^2 = U, \qquad \beta_v^2 V = 1 \tag{57a}$$

to reduce the value of this ratio to unity. We now observe that the more general transformation determined by

$$\alpha_u^2 = kU, \qquad \beta_v^2 V = k, \tag{57b}$$

where k is any non-vanishing constant, will accomplish the same result. Since, moreover, the curvature of the focal surfaces is positive, the asymptotic lines are imaginary. According to (14a) this implies that $d : m$ is positive for all values of u and v for which the ratio is defined at all; that is, the functions $U(u)$ and $V(v)$ have the same sign at all points of either focal surface. If both are positive, (57a) determines a real transformation. If both are negative we may use (57b) instead, where k is equated to -1, furnishing a *real* transformation in this case also.

Thus, *if the congruence has the properties Ia, Ib, and II, we may always assume that its equations are in the canonical form which we have been using, with the further specification that all of its coefficients are real functions of the real variables u and v.*

6. Final form for the differential equations of the congruence

We proved in Article 4 that the only congruences which possess properties Ia, II, III, and IV, are those for which

$$m_{\bar{u}} \neq 0, \qquad m_{\bar{v}} \neq 0, \qquad n_{\bar{u}} \neq 0, \qquad n_{\bar{v}} \neq 0, \qquad m_{\bar{u}\bar{v}} = n_{\bar{u}\bar{v}} = 0. \tag{58}$$

In all such cases we shall have

$$m = U_1 + V_1, \qquad n = U_2 + V_2, \tag{59}$$

where U_1 and U_2 are non-constant functions of $\bar{u}$, and V_1 and V_2 are non-constant functions of $\bar{v}$ alone. In particular, m and n are not equal to zero,

so that (17) shows that r also must be of the form $U + V$, say

$$r = U_3 + V_3. \tag{60}$$

Finally, the last of equations (17) furnishes the relation

$$2(U_1 + V_1)(U_2 + V_2) = i(U_3'' - V_3''), \tag{61}$$

or

$$2(U_1 V_2 + U_2 V_1) = iU_3'' - 2U_1 U_2 - iV_3'' - 2V_1 V_2. \tag{61a}$$

If we differentiate both members of this equation with respect to both $\bar{u}$ and $\bar{v}$, we find

$$U_1' V_2' + U_2' V_1' = 0,$$

or

$$\frac{U_2'}{U_1'} = -\frac{V_2'}{V_1'},$$

which would involve a contradiction if the common value of the two members were not a *constant*. We call this constant k, and note further that k must be different from zero, since U_1, U_2, V_1, V_2 are non-constant functions of their respective arguments. Consequently we find

$$U_2 = kU_1 + l_1, \qquad V_2 = -kV_1 + l_2, \qquad k \neq 0, \tag{62}$$

where k, l_1, and l_2 are constants.

If we substitute these values in (61a), we find

$$iU_3'' - 2kU_1^2 - 2(l_1 + l_2)U_1 = iV_3' - 2kV_1^2 + 2(l_1 + l_2)V_1 = a, \tag{63}$$

where a also must be a constant.

We have found

$$m = U_1 + V_1, \qquad n = k(U_1 - V_1) + l_1 + l_2.$$

Let us put

$$U_1 = \overline{U}_1 + \epsilon, \qquad V_1 = \overline{V}_1 - \epsilon,$$

where ϵ is a constant. Then

$$m = \overline{U}_1 + \overline{V}_1, \qquad n = k(\overline{U}_1 - \overline{V}_1) + 2k\epsilon + l_1 + l_2.$$

Since k is different from zero, we may choose ϵ subject to the condition

$$2k\epsilon + l_1 + l_2 = 0,$$

giving

$$m = \overline{U}_1 + \overline{V}_1, \qquad n = k(\overline{U}_1 - \overline{V}_1).$$

Let us write again U_1 and V_1 in place of $\overline{U}_1$ and $\overline{V}_1$. We shall have

$$\begin{gathered} m = U_1 + V_1, \qquad n = k(U_1 - V_1), \qquad k \neq 0, \\ iU_3'' - 2kU_1^2 = iV_3'' - 2kV_1^2 = a. \end{gathered} \tag{64}$$

If our congruence possesses property Ib, we may, according to Article 5, regard u and v as real variables and all of the coefficients, $m, n, a, b, \cdots, d'$, of our differential equations as real functions of u and v. In order to be able to draw conclusions from this remark, let us separate the functions U_1 and V_1 into their real and imaginary parts, putting

$$U_1 = U_{11} + U_{12}\, i, \qquad V_1 = V_{11} + V_{12}\, i.$$

If m is real, V_{12} must be equal to $-U_{12}$. Moreover U_{12} can not be equal to zero identically. For, from $U_{12} \equiv 0$ would follow $U_1 = \text{const.}$, since a function of the complex variable $\bar{u}$ can have an identically vanishing imaginary component only if it reduces to a real constant. Then V_1 would have to be a constant also. But both of these conclusions are contrary to our assumption, contained in Ia, that the focal surfaces are not ruled. Thus we find

$$U_1 = U_{11} + U_{12}\, i, \qquad V_1 = V_{11} - U_{12}\, i, \qquad U_{12} \neq 0,$$

and

$$n = k\,(U_{11} - V_{11} + 2U_{12}\, i).$$

Since $U_{11} + U_{12}\, i = U_1$ is a function of $\bar{u} = u + iv$, we have

(65a) $$(U_{11})_u = (U_{12})_v, \qquad (U_{12})_u = -(U_{11})_v,$$

and since $V_{11} - U_{12}\, i$ is a function of $\bar{v} = u - iv$,

(65b) $$(V_{11})_u = (U_{12})_v, \qquad (U_{12})_u = -(V_{11})_v.$$

From these equations we obtain

$$(U_{11} - V_{11})_u = (U_{11} - V_{11})_v = 0.$$

Therefore

$$U_{11} - V_{11} = l$$

must be a constant. Thus we have found

(66) $$n = k\,(l + 2U_{12}\, i),$$

where l is a real constant. If n is a *real* function of u and v, the same thing will be true of the functions

(67) $$n_u = 2\,(U_{12})_u\, ki, \qquad n_v = 2\,(U_{12})_v\, ki.$$

Now U_{12} can not be a constant; otherwise, according to (65a), U_{11} would also be a constant, and we should strike again the excluded case $m = \text{const.}$, in which one focal sheet is ruled. Since $(U_{12})_u$ and $(U_{12})_v$ are real functions of u and v, at least one of which is not equal to zero, and since n_u and n_v are also real functions, we conclude from (67) that k must be purely imaginary, so that we may write

$$k = ik', \qquad k' \gtrless 0.$$

If we insert this value of k in (66) we see that l must be equal to zero in order that n may be real. Thus we may write

$$m = U_1 + V_1, \qquad n = ik'(U_1 - V_1), \qquad k' \gtrless 0,$$

where U_1 and V_1 now indicate *conjugate* complex functions of $\bar{u}$ and $\bar{v}$ respectively.

Let us now transform our system of differential equations by the most general transformation of form (3) which satisfies the conditions (9). The new coefficients, $\bar{m}$ and $\bar{n}$, will be given by

$$\bar{m} = \frac{c_2}{\pm c_1 c_3} m, \qquad \bar{n} = \frac{c_1}{c_2 c_3} n,$$

and therefore we find

$$\bar{m} + i\bar{n} = \frac{1}{c_3}\left[\pm \frac{c_2}{c_1}(U_1 + V_1) - \frac{c_1}{c_2} k'(U_1 - V_1) \right],$$

$$\bar{m} - i\bar{n} = \frac{1}{c_3}\left[\pm \frac{c_2}{c_1}(U_1 + V_1) + \frac{c_1}{c_2} k'(U_1 - V_1) \right].$$

We can always satisfy the condition

$$\pm \frac{c_2}{c_1} + \frac{c_1}{c_2} k' = 0, \qquad \text{or} \qquad \left(\frac{c_2}{c_1}\right)^2 = \mp k',$$

by real values of $c_2 : c_1$ since we may use the minus sign when $k' < 0$ and the plus sign when k' is positive. If we choose $c_2 : c_1$ in this way, $\bar{m} + i\bar{n}$ and $\bar{m} - i\bar{n}$ will become functions of $\bar{u}$ alone, and of $\bar{v}$ alone, respectively, differing from U_1 and V_1 only by a common real constant factor. Let us denote these functions by U and V, and let us again change our notation by using m and n in place of $\bar{m}$ and $\bar{n}$. We shall have

(68) $$m + in = U, \qquad m - in = V,$$

and therefore

(69) $$m = \tfrac{1}{2}(U + V), \qquad n = \frac{1}{2i}(U - V),$$

where U and V are conjugate complex functions of $\bar{u}$ and $\bar{v}$ respectively.

The condition (63), or what amounts to the same thing, the last equation of (17), reduces to

$$U_3'' + \tfrac{1}{2}U^2 = V_3'' + \tfrac{1}{2}V^2 = \alpha,$$

where α must be a constant, and therefore

(70) $$U_3'' = \alpha - \tfrac{1}{2}U^2, \qquad V_3'' = \alpha - \tfrac{1}{2}V^2.$$

The remaining coefficients of our system of differential equations are given

by (11). If we make use of (11), (69), and (70), we find the following values

$$
(71)\qquad \begin{aligned} a &= 2\alpha + n^2 - m^2, & b &= -\,m_v, & c &= 0, & d &= m, \\ a' &= -\,n_u = +\,m_v, & b' &= -\,a, & c' &= n, & d' &= 0, \end{aligned}
$$

where, on account of (68) and (69),

$$
(72)\qquad m_u = n_v, \qquad m_v = -\,n_u.
$$

Since a should be real, it is clear moreover that α must be a *real* constant. We proceed to show that we may assume more specifically $\alpha \geqq 0$.

We are studying a system of differential equations of form (1) with the coefficients (69) and (71). Let us transform that system by putting

$$
(73)\qquad y = z_1, \qquad z = y_1, \qquad u = v_1, \qquad v = u_1,
$$

and denote the corresponding coefficients of the resulting system by m_1, n_1, etc. Then we shall find

$$
(71a)\qquad \begin{aligned} & m_1 = n, \qquad n_1 = m, \\ & a_1 = b' = -\,2\alpha + n_1^2 - m_1^2, \qquad b_1 = a' = -\,(m_1)_{v_1}, \\ & c_1 = d' = 0, \qquad d_1 = c' = n, \\ & a_1' = b = -\,(n_1)_{u_1}, \qquad b_1' = a = -\,a_1, \\ & c_1' = d = n_1, \qquad d_1' = c = 0, \end{aligned}
$$

with the conditions

$$
(72a)\qquad (n_1)_{v_1} = (m_1)_{u_1}, \qquad (n_1)_{u_1} = -\,(m_1)_{v_1}.
$$

The transformation (73) does not change the congruence; it merely interchanges the two focal sheets and the two families of developables. The two systems of coefficients (71) and (71a) exhibit the same structure, and may serve equally well for the purposes of studying the properties of the congruence. But these two systems differ in the sign of the constants which appear in a and a_1. We agree to retain (71) if α is positive, and to use (71a) if α is negative. Thus *we may assume, that α is not negative,* without thereby restricting the generality of our discussion.

The transformation of form (9) which we made was not uniquely determined. In fact we only determined the ratio of c_2 to c_1, and left c_3 as well as c_1 arbitrary. Let us now make a new transformation of form (9), but put $c_1 = c_2 = 1$ so as not to disturb the simplifications already effected. We find

$$
\bar{m} = \frac{m}{c_3}, \qquad \bar{n} = \frac{n}{c_3}, \qquad \bar{a} = \frac{a}{c_3^2} = \frac{2\alpha + n^2 - m^2}{c_3^2},
$$

and if we write

$$
\bar{a} = 2\bar{\alpha} + \bar{n}^2 - \bar{m}^2,
$$

we conclude

$$\bar{\alpha} = \frac{\alpha}{c_3^2}.$$

If $\alpha > 0$ we may therefore choose c_3 as a real number, in such a way as to reduce $\bar{\alpha}$ to any convenient positive value, for instance, the value 2. If α is zero, $\bar{\alpha}$ is zero likewise.

We have proved the following theorem.

THEOREM. *If a system of form* (1) *defines a congruence, which possesses the properties Ia, Ib, II, III, and IV, its coefficients are expressible in one of the two forms*

$$\begin{aligned} a &= 4 + n^2 - m^2, & b &= -m_v, & c &= 0, & d &= m, \\ a' &= -n_u, & b' &= -a, & c' &= n, & d' &= 0, \end{aligned} \tag{74}$$

where

$$m_u = n_v, \qquad m_v = -n_u, \tag{75}$$

or else

$$\begin{aligned} a &= n^2 - m^2, & b &= -m_v, & c &= 0, & d &= m, \\ a' &= -n_u, & b' &= -a, & c' &= n, & d' &= 0, \end{aligned} \tag{74$'$}$$

where again

$$m_u = n_v, \qquad m_v = -n_u. \tag{75$'$}$$

We shall speak of these two forms as *form A* and *form B* respectively.

7. Relation to the theory of functions of form *A*

The developables of the congruence, determined by a relation of the form $w = F(z)$, where z and w are two complex variables on the Riemann sphere, are obtained by equating α and β to constants, where

$$t = \alpha + i\beta = i \int \frac{\sqrt{w'}\, dz}{z - w}.* \tag{76}$$

The differential equations of such a congruence constitute a system of form (1) whose coefficients are exactly of the form (74), conditioned by (75), the independent variables being α and β instead of u and v. Moreover $m + in$, which must on account of (75) be a function of $t = \alpha + i\beta$, which variable corresponds to the $\bar{u}$ of the present paper, is connected with the relation $w = F(z)$ by the equation

$$m + in = -\frac{1}{\sqrt{w'}} \left[1 + w' + \frac{1}{2}(z - w) \frac{w''}{w'} \right]. \tag{77}$$

We wish to show that conversely, when $m + in$ is given as any analytic

* *Line geometric representations*, p. 285.

function of t, w can be determined as a function of z so as to satisfy this equation; moreover we shall investigate what analytical processes are actually required for this purpose, and to what extent the function $w = F(z)$ is actually determined.

In equations (76) and (77), w' and w'' indicate first and second order derivatives with respect to z. Let $m + in$ be given as a function of $t = \alpha + i\beta$, say

$$m + in = \phi(t), \tag{78}$$

and let us denote by w_1, w_2, w_3 first, second, and third order derivatives of w with respect to t. We find, making use of (76),

$$w' = w_1 \frac{i\sqrt{w'}}{z - w}, \qquad \frac{dz}{dt} = \frac{z - w}{i\sqrt{w'}} = -\frac{(z - w)^2}{w_1},$$

and therefore

$$\sqrt{w'} = \frac{iw_1}{z - w}, \qquad w' = \frac{-w_1^2}{(z - w)^2}. \tag{79}$$

We find further

$$\frac{w''}{w'} = \left(\frac{w_2}{w_1} - 2\frac{\frac{dz}{dt} - w_1}{z - w}\right)\frac{i\sqrt{w'}}{z - w} \tag{80}$$

$$= -\frac{2}{(z - w)^2}\left(w_2 + \frac{w_1^2}{z - w}\right) - \frac{2}{z - w},$$

and consequently

$$1 + w' + \frac{1}{2}(z - w)\frac{w''}{w'} = \frac{-2w_1}{(z - w)^2} - \frac{w_2}{z - w}, \tag{81}$$

so that (77) and (78) give

$$\frac{w_2}{w_1} + \frac{2w_1}{z - w} = i(m + in) = i\phi(t), \tag{82a}$$

or

$$\frac{w_2}{w_1^2} + \frac{2}{z - w} = \frac{i\phi(t)}{w_1}. \tag{82b}$$

Let us differentiate both members of (82b) with respect to t. We find

$$\frac{w_3}{w_1^2} - 2\frac{w_2^2}{w_1^3} - \frac{2}{(z - w)^2}\left[-\frac{(z - w)^2}{w_1} - w_1\right] = \frac{i\phi'(t)}{w_1} - \frac{i\phi(t)\,w_2}{w_1^2},$$

or

$$\frac{w_3}{w_1} - 2\frac{w_2^2}{w_1^2} + 2 + \frac{2w_1^2}{(z - w)^2} = i\phi'(t) - i\phi(t)\frac{w_2}{w_1}.$$

Into this equation let us substitute for $z - w$, the value obtained for it from

(82). We find finally

$$\frac{w_3}{w_1} - \frac{3}{2}\frac{w_2^2}{w_1^2} = \{w, t\} = i\phi'(t) + \frac{1}{2}\phi(t)^2 - 2, \tag{83}$$

a Schwarzian differential equation for w as function of t. This equation being integrated, we find from (82),

$$\frac{2}{z - w} = \frac{i\phi(t)}{w_1} - \frac{w_2}{w_1^2}, \tag{84}$$

an equation enabling us to determine z as a function of t without any further integration. The relation $w = F(z)$ must then be found by eliminating t.

Of course the essential part of the integration of (83) consists in integrating the *Riccati* equation for $\eta = w_2/w_1$ to which (83) reduces.

It is well known that, if w is a particular solution of (83), the most general solution will be

$$\bar{w} = \frac{\alpha w + \beta}{\gamma w + \delta}, \qquad \alpha\delta - \beta\gamma \neq 0, \tag{85a}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary constants. Of course we have the relation

$$\frac{2}{\bar{z} - \bar{w}} = \frac{i\phi(t)}{\bar{w}_1} - \frac{\bar{w}_2}{\bar{w}_1^2}$$

corresponding to (84). If we substitute for $\bar{w}$ the value just found, we find

$$\bar{z} = \frac{\alpha z + \beta}{\gamma z + \delta}. \tag{85b}$$

Moreover, it is not difficult to verify directly that $\bar{w}$ and $\bar{z}$ as given by (85*a*) and (85*b*) will satisfy the equation $m + in = \phi(t)$ if w and z do. In other words, *the right member of* (77) *is a differential invariant of the three-parameter group*

$$\bar{z} = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \bar{w} = \frac{\alpha w + \beta}{\gamma w + \delta}, \tag{85}$$

which moreover preserves its form when the variables z and w are interchanged. The integral t, given by (76) *is an integral invariant belonging to the same group.*

This symmetry leads us to conclude that z should satisfy a Schwarzian equation similar to (83). In fact we have

$$z_1 w_1 = -(z - w)^2,$$

and therefore, if we make use of (82*a*),

$$\frac{z_2}{z_1} = -i\phi(t) + \frac{2z_1}{z - w},$$

whence, by another differentiation and simple reductions,

$$\{z, t\} = -i\phi'(t) + \tfrac{1}{2}\phi(t)^2 - 2. \tag{86}$$

Thus we see that *the integration of either of the two Schwarzian equations* (83) *or* (86) *carries with it the integration of the other.*

Our principal result may be formulated as follows: *To every system of differential equations of form* (1), *whose coefficients satisfy the relations* (74) *and* (75), *corresponds a family of analytic functions depending upon three arbitrary complex constants. If $w = F(z)$ is one of these, the most general function of the family will be defined by*

$$\frac{\alpha w + \beta}{\gamma w + \delta} = F\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right). \tag{87}$$

Since the congruences which correspond to the ∞^3 functions of such a family belong to the same system of form (1), *these congruences are projectively equivalent. They may all be obtained from any one of them by means of the projective transformations of the six-parameter group which leaves the Riemann sphere invariant, the three complex parameters being equivalent to six real parameters.*

Of course all of these congruences have the properties Va and Vb as well as those which have been mentioned heretofore. The corresponding system of differential equations, however, is satisfied also by other congruences, projectively equivalent to those mentioned, whose directrix quadrics do *not* coincide with the Riemann sphere, and to these there do not correspond any analytic functions.

To complete the proof that our list of properties is characteristic of the class of congruences defined in this way by non-linear functional relations, it remains to show that a system whose coefficients satisfy relations (74)′ and (75)′ which, as we have seen, can not be reduced to the form characterized by (74) and (75), is excluded by property Va. We shall show that such congruences correspond to the case of degenerate directrix quadrics.

8. Discussion of form B

For a large part of this discussion it will be advisable to consider forms A and B together, by writing

$$\begin{aligned} &a = 2k + n^2 - m^2, \quad && b = -m_v, \quad && c = 0, \quad && d = m, \\ &a' = -n_u, && b' = -a, && c' = n, && d' = 0, \\ &&& m_u = n_v, \quad m_v = -n_u. \end{aligned} \tag{88}$$

Form A corresponds to $k = 2$, and Form B to $k = 0$.

As in Article 4, we consider three local coördinate systems, those determined

by the two focal surfaces, and that of the congruence. The transformation equations (44) and (46) may now be simplified slightly by making use of the relations

$$(89)\quad m_{uu} + m_{vv} = n_{uu} + n_{vv} = 0, \qquad r_{uu} = 2k - m^2 + n^2, \qquad r_{vv} = -r_{uu},$$

which were not presupposed in Article 4.

The equation of the osculating quadric Q_y of S_y, at the point of S_y which corresponds to the values u and v of the parameters, is

$$x_1^{(1)} x_4^{(1)} - x_2^{(1)} x_3^{(1)} + 2a_1' b_1 x_4^{(1)^2} = 0,^*$$

when referred to the local coördinate system of S_y. The osculating quadric Q_z of S_z at the corresponding point of S_z, is given by

$$x_1^{(2)} x_4^{(2)} - x_2^{(2)} x_3^{(2)} + 2a_2' b_2 x_4^{(2)^2} = 0.$$

We refer both of these quadrics to the local coördinate system of the congruence. We find the following equations; for Q_y:

$$(90a)\quad \begin{aligned} &x_2^2 + m^2 x_3^2 + \left(2k - m^2 + n^2 + \frac{n_v^2}{n^2}\right) x_4^2 \\ &\qquad - 2mx_1 x_4 - 2\frac{n_v}{n} x_2 x_4 + 2n_v x_3 x_4 = 0, \end{aligned}$$

and for Q_z

$$(90b)\quad \begin{aligned} &x_1^2 + n^2 x_4^2 - \left(2k - m^2 + n^2 - \frac{m_u^2}{m^2}\right) x_3^2 \\ &\qquad - 2\frac{m_u}{m} x_1 x_3 - 2nx_2 x_3 + 2m_u x_3 x_4 = 0. \end{aligned}$$

These equations may also be written as follows:

$$(91)\quad \begin{aligned} Q_y &\equiv \left(x_2 - \frac{n_v}{n} x_4\right)^2 - 2m\left(x_1 - \frac{m_u}{m} x_3\right) x_4 \\ &\qquad + m^2 (x_3^2 - x_4^2) + (2k + n^2) x_4^2 = 0, \\ Q_z &\equiv \left(x_1 - \frac{m_u}{m} x_3\right)^2 - 2n\left(x_2 - \frac{n_v}{n} x_4\right) x_3 \\ &\qquad - n^2 (x_3^2 - x_4^2) - (2k - m^2) x_3^2 = 0. \end{aligned}$$

According to our general theory, these quadrics must intersect in four straight lines (at least in the case $k = 2$), namely in the four generators of the directrix quadric which correspond to the line of the congruence under consideration. We wish to confirm this fact, and besides find the actual equations of these four lines of intersection. For this purpose we replace the quadrics Q_y and Q_z,

* *Second Memoir*, p. 82.

by the quadrics $Q_z + Q_y$ and $Q_z - Q_y$ of their pencil. We find

$$(92)\quad \begin{aligned} Q_z + Q_y &\equiv \left(x_1 - \frac{m_u}{m}x_3 - m\,x_4\right)^2 + \left(x_2 - \frac{n_v}{n}x_4 - nx_3\right)^2 \\ &\qquad + 2\,(m^2 - n^2 - k)\,(x_3^2 - x_4^2) = 0, \\ Q_z - Q_y &\equiv \left(x_1 - \frac{m_u}{m}x_3 + mx_4\right)^2 - \left(x_2 - \frac{n_v}{n}x_4 + nx_3\right)^2 \\ &\qquad - 2k\,(x_3^2 + x_4^2) = 0. \end{aligned}$$

Let us put

$$(93)\qquad x_1' = x_1 - \frac{m_u}{m}x_3 + mx_4, \qquad x_2' = x_2 + nx_3 - \frac{n_v}{n}x_4.$$

Then

$$(94)\quad \begin{aligned} Q_z + Q_y &\equiv (x_1' - 2mx_4)^2 + (x_2' - 2nx_3)^2 \\ &\qquad + 2\,(m^2 - n^2 - k)\,(x_3^2 - x_4^2) = 0, \\ Q_z - Q_y &\equiv (x_1')^2 - (x_2')^2 - 2k\,(x_3^2 + x_4^2) = 0. \end{aligned}$$

We find that these quadrics have indeed four lines of intersection, namely the two lines

$$(95a)\qquad x_1' - x_2' = 2k\lambda_r\,(x_3 + ix_4), \qquad x_1' + x_2' = \frac{1}{\lambda_r}\,(x_3 - ix_4) \quad (r = 1, 2),$$

where λ_1 and λ_2 are the two roots of the quadratic

$$(95b)\qquad k\lambda - \frac{1}{2\lambda} = -\,(n + mi),$$

and the two lines

$$(95c)\qquad x_1' - x_2' = 2k\mu_r\,(x_3 - ix_4), \qquad x_1' + x_2' = \frac{1}{\mu_r}\,(x_3 + ix_4) \quad (r = 1, 2),$$

where μ_1 and μ_2 are roots of

$$(95d)\qquad k\mu - \frac{1}{2\mu} = -\,n + mi.$$

These formulæ become indeterminate in the case $k = 0$, which interests us primarily. The formulæ for that case however are much simpler. We have

$$\begin{aligned} Q_z + Q_y &\equiv (x_1' - 2mx_4)^2 + (x_2' - 2nx_3)^2 + 2\,(m^2 - n^2)\,(x_3^2 - x_4^2) = 0, \\ Q_z - Q_y &\equiv (x_1')^2 - (x_2')^2 = 0, \end{aligned}$$

and obtain the following equations for the four lines of intersection:

$$(96)\quad \begin{aligned} &(l_1)\quad x_2' - (n - im)\,x_3 - (m + in)\,x_4 = 0, \qquad x_1' - x_2' = 0, \\ &(l_2)\quad x_2' - (n + im)\,x_3 - (m - in)\,x_4 = 0, \qquad x_1' - x_2' = 0, \\ &(l_3)\quad x_2' - (n - im)\,x_3 + (m + in)\,x_4 = 0, \qquad x_1' + x_2' = 0, \\ &(l_4)\quad x_2' - (n + im)\,x_3 + (m - in)\,x_4 = 0, \qquad x_1' + x_2' = 0. \end{aligned}$$

The lines l_1 and l_2 are in the plane π_1, whose equation is

$$x_1' - x_2' = x_1 - x_2 - \left(n + \frac{m_u}{m}\right)x_3 + \left(m + \frac{n_v}{n}\right)x_4 = 0, \tag{97a}$$

while l_3 and l_4 are in the plane π_2, whose equation is

$$x_1' + x_2' = x_1 + x_2 + \left(n - \frac{m_u}{m}\right)x_3 + \left(m - \frac{n_v}{n}\right)x_4 = 0. \tag{97b}$$

We shall prove that these two planes, π_1 and π_2, do not vary with u and v, and that, consequently, the four lines $l_1, \cdots, l_4$, associated in this way with every line of the congruence, remain in two fixed planes when u and v vary over their ranges.

The coördinates of the point of intersection of l_1 and l_2 may be found from (96) and (93). Thus we find the expression

$$\alpha = n\left(n + \frac{m_u}{m}\right)y + m\left(m + \frac{n_v}{n}\right)z + n\rho + m\sigma \tag{98a}$$

for this point, the coefficients of y, z, ρ, and σ being proportional to the coördinates of the point. The expressions

$$\beta = iny + \left(m + in + \frac{n_v}{n}\right)z + \sigma, \tag{98b}$$

$$\gamma = \left(n + im + \frac{m_u}{m}\right)y + imz + \rho, \tag{98c}$$

represent two other points of the plane π_1, one being on l_1 and one on l_2, both different from α. These three points are therefore not collinear and we may think of π_1 as being determined by these three points.

We find

$$\begin{aligned} \alpha_u = &\left(2nn_u + n_u\frac{m_u}{m} + n^2\frac{m_u}{m} + mn_v + n^3\right)y \\ &+ \left(2mm_u + m_u\frac{n_v}{n} - nm_v + mn_v + m^2 n\right)z \\ &+ (n^2 + n_u)\rho + (m_u + mn)\sigma, \\ \alpha_v = &\left(2nn_v + n_v\frac{m_u}{m} + mn^2 - mn_u + nm_u\right)y \\ &+ \left(2mm_v + m_v\frac{n_v}{n} + nm_u + m^2\frac{n_v}{n} + m^3\right)z \\ &+ (n_v + mn)\rho + (m_v + m^2)\sigma, \end{aligned} \tag{99}$$

$$
(99)\quad
\begin{aligned}
\beta_u &= \left(in_u + in\frac{m_u}{m} + mn + in^2 + n_v\right)y + (m_u + in_u + mn)z + in\rho,\\
\beta_v &= \left(in_v - n_u + n\frac{m_u}{m}\right)y\\
&\qquad + \left(imn + m_v + 2in_v + m\frac{n_v}{n} + m^2 - n^2\right)z\\
&\qquad\qquad + n\rho + (m + in)\sigma,\\
\gamma_u &= \left(n_u + 2im_u + n\frac{m_u}{m} + imn + n^2 - m^2\right)y\\
&\qquad + \left(im_u - m_v + m\frac{n_v}{n}\right)z + (n + im)\rho + m\sigma,\\
\gamma_v &= (n_v + im_v + mn)y\\
&\qquad + \left(m_u + im_v + im\frac{n_v}{n} + mn + im^2\right)z + im\sigma.
\end{aligned}
$$

Each of these expressions represents a point whose local coördinates x_1, x_2, x_3, x_4 are proportional to the coefficients of y, z, ρ, and σ. If these coördinates are substituted in (97a), we find that this equation is satisfied by all of them. Consequently for all possible variations of u and v, the three points α, β, γ remain in a fixed plane; in other words the plane π_1 is a fixed plane. The same thing may be proved in the same fashion of the plane π_2.

In the general case ($k > 0$), the locus described by the four lines $l_1, \cdots, l_4$ is a quadric, the directrix quadric. Thus it appears that the case $k = 0$ corresponds to the case of a degenerate directrix quadric. But we have not yet proved this in conclusive fashion. For, while we know that the four lines of intersection of the osculating quadrics Q_y and Q_z are at the same time generators of the directrix quadric when $k > 0$, we have not actually shown this to be the case also when $k = 0$.

To supply this proof we might use the method of limits. More directly we may argue as follows. The lines l_1, l_2, l_3, l_4 were defined as the lines common to Q_y and Q_z, the osculating quadrics of S_y and S_z at two corresponding points, P_y and P_z. These lines form a skew quadrilateral. Let R_1 be an osculating ruled surface of S_y, made up of the asymptotic tangents (of the first kind) of S_y along the fixed asymptotic curve of the second kind which passes through P_y. The generator g of R_1 which passes through P_y will be a generator of Q_y also, of the same kind as l_1 and l_3, for example. Let P_y move along the asymptotic curve of the second kind, the curve of contact between R_1 and S_y. Then l_1 and l_2 can move only in π_1, and l_3 and l_4 only in π_2. The lines l_2 and l_4 intersect g and are asymptotic tangents of the ruled surface R_1. Since l_2 can move only in π_1, if it moves at all, it will have an envelope

which would be a plane asymptotic curve of R_1. But a plane curve can be an asymptotic curve only when it reduces to a straight line. Therefore l_2, and similarly l_4 will have to remain fixed. In other words, l_2 and l_4 will be the straight directrices of the ruled surface R_1. Similarly we prove l_1 and l_3 to be the remaining two directrices.

Finally this fact may be proved analytically as well, by setting up the differential equations of the osculating ruled surfaces and determining the flecnode tangents of these surfaces; but we shall refrain from any further discussion of this question, as we have now accomplished the main purpose of this paper. We have seen that form B corresponds to the case of a degenerate directrix quadric, and is therefore excluded by property Va.

Thus, *properties Ia, Ib, II, III, IV, Va, and Vb are characteristic of the class of congruences defined by a functional relation between two complex variables on the same sphere, provided that this relation be non-linear.*

Incidentally it will be noted that a large part of property Va is a consequence of the properties which precede. It is merely the non-degeneracy of the directrix quadric which requires specific formulation.

9. The case of a linear function

If the relation between w and z is bilinear, the properties of the corresponding congruence are, in some respects, essentially different from those which hold in the general case. In the extremely special case when this relation reduces to $z =$ const., or $w =$ const., the congruence evidently reduces to a bundle of lines. In all other cases we may write

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \alpha\delta - \beta\gamma = 1, \tag{100}$$

and we may choose the ambiguous symbol $\sqrt{w'}$ in such a way as to make

$$\sqrt{w'} = \frac{1}{\gamma z + \delta}. \tag{101}$$

If the bilinear relation reduces to $w = z$, the congruence becomes completely indeterminate, and may be thought of as being replaced by the complex of tangents of the Riemann sphere. Let us exclude this case also; *the differential equations, as determined for the general case, remain valid whenever the bilinear relation between z and w does not reduce to*

$$z = \text{const.}, \qquad w = \text{const.}, \qquad \textit{or} \qquad w = z.$$

The coefficients, given by

$$
(102)\quad
\begin{array}{llll}
m = -\frac{1}{2}(\alpha + \alpha_0 + \delta + \delta_0), & & n = -\dfrac{1}{2i}(\alpha - \alpha_0 + \delta - \delta_0), & \\
a = 4 + n^2 - m^2, & b = 0, & c = 0, & d = m, \\
a' = 0, & b' = -a, & c' = n, & d' = 0,
\end{array}
$$

are all real constants.

Consequently both focal sheets are ruled surfaces and more specifically *quadrics*, since both families of asymptotic lines upon each of them will be composed of straight lines. Moreover, either or both of the focal sheets may degenerate into straight lines, since either m or n, or both m and n may vanish. The congruence will be a W-congruence, as in the general case, and its developables will still determine isothermally conjugate systems of curves on the focal sheets, except in the cases when these sheets degenerate. The focal quadrics or lines will be real, as in the general case, and whenever non-degenerate, the focal quadrics will be surfaces of positive curvature.

The asymptotic lines of the focal surfaces, being straight, of course belong to linear complexes. But a straight line belongs to infinitely many linear complexes, and consequently the directrices mentioned in property IV become indeterminate. A similar situation arises in connection with the directrix quadric. Thus while, in the case of a linear function, only slight modifications are required for properties Ia, Ib, II, and III, properties IV and V become meaningless, and must either be replaced by others or else omitted.

Let a and b be the two roots, assumed to be distinct, of the quadratic equation

$$
(103)\qquad -\gamma z^2 + (\alpha - \delta)z + \beta = 0,
$$

obtained by equating w to z. Then we may replace (100) by

$$
(104)\qquad \frac{w-a}{w-b} = K\frac{z-a}{z-b}, \qquad K \neq 0, \qquad K \neq 1.
$$

We have seen in Article 7 that the projective properties of the congruence are not altered if we replace w and z by linear functions of w and z with the *same* constant coefficients. We may therefore study, in place of (104), the far simpler relation

$$
(105)\qquad w = Kz,
$$

without any essential loss of generality. We shall prefer to write

$$
(106)\qquad w = k^2 z,
$$

rather than (105), k being one of the two square roots of K determined in any way that may be convenient.

We determine the ambiguous symbol $\sqrt{w'}$ by the equation

$$\sqrt{w'} = k,$$

and $\sqrt{w'_0}$ by the relation

$$\sqrt{w'}\,\sqrt{w'_0} = \sqrt{w'\,w'_0} = kk_0 > 0,$$

obtaining

$$m + in = -\left(k + \frac{1}{k}\right), \qquad m - in = -\left(k_0 + \frac{1}{k_0}\right). \tag{107}$$

By making use of the general formulæ* we find the cartesian equations

$$\left(\frac{kk_0 + 1}{k + k_0}\right)^2 (\xi^2 + \eta^2) + \zeta^2 = 1, \tag{108}$$

and

$$\left(\frac{kk_0 - 1}{k - k_0}\right)^2 (\xi^2 + \eta^2) + \zeta^2 = 1, \tag{109}$$

for the focal quadrics. It is evident that (108) will, in general, be an ellipsoid, and (109) a hyperboloid of two sheets. If we put

$$k = k_1 + k_2\,i, \qquad k_0 = k_1 - k_2\,i, \tag{110}$$

we may write

$$\frac{(k_1^2 + k_2^2 + 1)^2}{4k_1^2}(\xi^2 + \eta^2) + \zeta^2 = 1 \tag{111}$$

for the focal ellipsoid, and

$$-\frac{(k_1^2 + k_2^2 - 1)^2}{4k_2^2}(\xi^2 + \eta^2) + \zeta^2 = 1 \tag{112}$$

for the focal hyperboloid. Both are surfaces of revolution, with the ζ-axis as axis; they touch each other and the Riemann sphere at the points $(0, 0, \pm 1)$.

Let r be the radius of the circular intersection of (111) with the $\xi\eta$-plane. Then

$$r = \left|\frac{2k_1}{k_1^2 + k_2^2 + 1}\right|, \tag{113}$$

and since

$$k_1^2 + 1 \geqq 2k_1,$$

we find

$$0 \leqq r < 1, \tag{114}$$

except in the case $k_1 = 1$, $k_2 = 0$ in which $k^2 = 1$, and which we have excluded from consideration. Thus the focal ellipsoid lies entirely inside of the Riemann

* *Line-geometric representations*, p. 285, equations (51). In this connection we wish to call attention to an error which occurs on this page. The two foci of a line coincide not only in the cases mentioned, but also whenever $z = F(z)$.

sphere, touching it at the points $(0, 0, \pm 1)$. It reduces to the segment on the ζ-axis between these two points, if $k_1 = 0$, that is, if $K = k^2$ is a negative real number, and can degenerate in no other way.

The $\xi\zeta$-plane intersects the hyperboloid of revolution (112) in a hyperbola whose semi-transverse axis is equal to unity. Its semi-conjugate axis is equal to

$$s = \left| \frac{2k_2}{k_1^2 + k_2^2 - 1} \right|. \tag{115}$$

If $k_1^2 + k_2^2 = 1$, $k_2 \neq 0$, the hyperboloid reduces to the pair of parallel planes $\zeta = \pm 1$, and may be replaced in its rôle as a focal sheet by their infinitely distant line of intersection. This corresponds to the case $|K| = 1$, $K \neq 1$, of a non-identical elliptic substitution which represents a rotation of the sphere around the ζ-axis.

If $k_2 = 0$, $k_1^2 \neq 1$, the hyperboloid reduces to

$$\xi^2 + \eta^2 = 0,$$

or more properly to that (projective) segment of the ζ-axis exterior to the sphere. In this case the substitution is hyperbolic, K being real and positive but different from unity.

If $k_1 = 0$, $k_2^2 \doteq 1$, both focal quadrics degenerate. In this case $K = -1$, and the relation $w = -z$ represents a rotation of the sphere through 180°.

The developables of the congruence are obtained by equating to constants the variables α and β, defined by

$$t = \alpha + i\beta = i \int \frac{\sqrt{w'}\, dz}{z - w} = \frac{ik}{1 - k^2} \log z. \tag{115}$$

Consequently the images of these developables form, in general, an isothermal orthogonal system of logarithmic spirals in the plane $\zeta = 0$, or a similar system of loxodromes on the sphere. Only in the special cases already enumerated, when $ik/(1 - k^2)$ reduces to a real or a purely imaginary constant, do these loxodromes reduce to circles.

If the two united points of the transformation (100) coincide, so that (103) has coincident roots, the transformation may be reduced to the form

$$w = z + l.$$

But we may simplify this further, replacing z and w by kz and kw simultaneously, and then equating k to a real multiple of l. We shall choose the canonical form

$$w = z + 2. \tag{116}$$

We find in this way a focal ellipsoid

$$2(\xi^2 + \eta^2) + 4(\zeta - \tfrac{1}{2})^2 = 1, \tag{117}$$

with center at $(0, 0, \frac{1}{2})$, and semi-axes $1/\sqrt{2}$, $1/\sqrt{2}$, $1/2$, touching the Riemann-sphere at $(0, 0, 1)$, and passing through the origin. The second sheet of the focal surface is given by

$$\eta = 0, \qquad \zeta = 1, \tag{118}$$

the straight line parallel to the ξ-axis through the point $(0, 0, 1)$. In this case, of a parabolic transformation, we find

$$\alpha + i\beta = \frac{-i}{2} z. \tag{119}$$

Consequently the developables correspond to the lines parallel to the ξ- and η-axes of the $\xi\eta$-plane, and to the corresponding system of circles on the sphere.

We may summarize this discussion as follows:

If z and w are connected by a bilinear relation, which does not reduce to one of the exceptional forms

$$z = \text{const.}, \qquad w = \text{const.}, \qquad \textit{or} \qquad z = w,$$

the corresponding congruence may be transformed into one of the following six types, by means of a collineation which leaves the Riemann sphere invariant.

A. If the relation between z and w, interpreted as a transformation, is loxodromic, excepting the case when the multiplier K is real and negative, the focal surfaces consist of an ellipsoid and a two-sheeted hyperboloid of revolution, their common axis of revolution being a diameter of the Riemann sphere and their common center the center of the sphere. The focal ellipsoid lies entirely inside, and the focal hyperboloid entirely outside of the sphere. The two focal quadrics touch each other and the Riemann sphere at the two points in which their common axis pierces the sphere.

B. If the multiplier K is a negative real number, different from -1, the focal ellipsoid degenerates into that segment of the axis which lies inside of the sphere, while the focal hyperboloid remains proper.

C. If the multiplier K is equal to -1, the focal ellipsoid degenerates as in B; the focal hyperboloid reduces to a pair of planes, perpendicular to the axis at the end points of the segment into which the focal ellipsoid has degenerated, and may be replaced as a focal locus by the infinitely distant line of intersection of the two planes.

D. If the transformation is elliptic, that is if $|K| = 1$, the focal ellipsoid remains proper, except in the case $K = -1$ mentioned under C; but the focal hyperboloid may be replaced by the infinitely distant line of the planes perpendicular to the axis.

E. If the transformation is hyperbolic, that is if K is real and positive, but

different from unity, the focal ellipsoid remains proper, but the focal hyperboloid reduces to that portion of the axis which lies outside of the sphere.

F. If the transformation is parabolic, the focal ellipsoid touches the sphere at one point only, and the second focal sheet consists of a straight line, tangent to the sphere and the focal ellipsoid at their point of contact.

10. Congruences which possess properties I, II, III, except that one or both focal sheets may be ruled

In order that we may see the contents of Article 9 in proper perspective, we discuss some closely related, but slightly more general problems.

Let us modify property I*a*, by admitting that S_y may be a non-degenerate, and non-developable *ruled* surface, but retain properties II and III. Then we may still study our congruence by means of a system of form (1), whose coefficients satisfy the relations (11) and (12). We shall have besides

$$(120)\quad m \neq 0, \quad \text{and either} \quad m_{\bar u} = 0 \text{ or } m_{\bar v} = 0, \quad \text{or} \quad m_{\bar u} = m_{\bar v} = 0.$$

In all such cases we shall have, from (17),

$$m_{\bar u \bar v} = n_{\bar u \bar v} = r_{\bar u \bar v} = 0,$$

as in the case of non-ruled focal sheets.

Let us consider the case

$$(121)\qquad m_{\bar u} = 0, \quad m_{v} \neq 0,$$

in which

$$(122)\quad m = V_1(\bar v), \quad n = U_2(\bar u) + V_2(\bar v), \quad r = U_3(\bar u) + V_3(\bar v).$$

Then S_y is a ruled surface, not a quadric, and the generators of S_y are obtained by equating $\bar v$ to constants. The three conditions (17) reduce to

$$(123)\qquad 2V_1(U_2 + V_2) = i(U_3'' - V_3'').$$

If we differentiate both members of this equation with respect to both $\bar u$ and $\bar v$, we find

$$V_1' U_2' = 0,$$

and therefore $U_2' = 0$, since we are discussing the case $m_{\bar v} \neq 0$. Therefore we must have $n_{\bar u} = 0$. Thus S_z must also be a ruled surface, and its generators correspond to those of S_y. Such congruences actually exist. They correspond to the functions

$$(124)\quad m = V_1(\bar v), \quad n = V_2(\bar v), \quad r = U_3(\bar u) + V_3(\bar v),$$

where

$$(125)\qquad U_3''' = k, \quad V_3''' = k - 2iV_1 V_2,$$

k being an arbitrary constant. We have found the following result: *If a*

W-congruence, with distinct focal sheets, has a non-degenerate and non-developable ruled surface, not a quadric, as one of its focal sheets, and if the developables of the congruence determine an isothermally conjugate system of curves on that sheet, the second focal sheet will also be ruled, and the lines of the congruence will make the generators of the two focal sheets correspond.

In drawing this conclusion we have made no use of property Ib, which demands not only that the focal sheets shall be real, but also that each of them shall have a positive measure of curvature. Whenever a congruence possesses this property, we may assume, according to Article 6, that the variables u and v and all of the coefficients of the corresponding system of differential equations are *real*. In particular we may, therefore, assume that

$$m = V_1(\bar{v}) = V_1(u - iv)$$

is real for all real values of u and v. But this function of the complex variable $u - iv$ can have an identically vanishing imaginary part only if it reduces to a real constant, and in that case S_y will have to be a quadric, a result which is quite evident geometrically. Unless this quadric degenerates, we have $m \neq 0$, and

$$n = U_2(\bar{u}) + V_2(\bar{v}),$$

where n also must be *real*. We may therefore assume $U_2(\bar{u})$ and $V_2(\bar{v})$ to be conjugate complex functions of the conjugate complex variables $u + iv$ and $u - iv$.

If now we assume that S_z also is a real ruled surface of positive curvature, it also must be a quadric, and n also must be a real constant. According to (11) and (12), we have in this case

$$\begin{aligned} &a = r_{uu}, \quad && b = 0, \quad && c = 0, \quad && d = m, \quad && m \neq 0, \\ &a' = 0, && b' = r_{vv}, && c' = n, && d' = 0, && n \neq 0, \end{aligned}$$

where

$$r_{uu} + r_{vv} = 0, \qquad r_{uv} = 2mn.$$

The last two conditions give

$$r_{uu} = k, \qquad r_{vv} = -k,$$

where k is a constant, which we may assume to be real in accordance with Article 6. Thus we find

$$(126) \qquad \begin{aligned} &a = k, \quad && b = 0, \quad && c = 0, \quad && d = m, \quad && m \neq 0, \\ &a' = 0, && b' = -k, && c' = n, && d' = 0, && n \neq 0, \end{aligned}$$

where m, n, and k are real constants.

This system has the same form as (102), except that in the latter system we have the relation

$$a + m^2 - n^2 = 4,$$

which is not demanded by (126). Let us, however, make a transformation of form (3) restricted by (9). We obtain a new set of coefficients $\overline{m}$, $\overline{n}$, etc., also constants, where

$$\overline{m} = \frac{c_2}{\pm c_1 c_3} m, \qquad \overline{n} = \frac{c_1}{c_2 c_3} n, \qquad \overline{a} = \frac{a}{c_3^2},$$

so that

$$\overline{a} + \overline{m}^2 - \overline{n}^2 = \frac{1}{c_3^2}\left(a + m^2 \lambda^2 - \frac{n^2}{\lambda^2}\right) = \frac{m^2 \lambda^4 + a\lambda^2 - n^2}{c_3^2 \lambda^2},$$

where we have put

$$\lambda = \frac{c_2}{c_1}.$$

If a is positive, $\overline{a} + \overline{m}^2 - \overline{n}^2$ will be positive if λ be chosen as a real number sufficiently great, and this will be so even if m is equal to zero. Therefore we may, in this case, select λ and c_3 as real numbers so as to make $\overline{a} + \overline{m}^2 - \overline{n}^2$ equal to 4. If a is negative, we may change our notation, replacing y, z, u, v in order by z, y, v, u. This is equivalent to an interchange of m and n, and a and b'. Since b' will, in this case, be positive we see that the same reduction may be accomplished in this case also. If $a = 0$, $m \neq 0$, the same conclusion follows. If $a = m = 0$, $n \neq 0$, this transformation still remains possible, if we combine it with the operation of permuting y and z, and u and v, resulting in an interchange of m with n. Only in the case $a = m = n = 0$ does this reduction become impossible. It is easy to see that, in this last case, the congruence reduces to a linear congruence with real focal lines, real lines of the congruence passing through every point of each focal line. In contrast with this, we might speak of case C of Article 9 as an *incomplete* linear congruence, since in case C one of the focal loci is not a complete line, but merely a *segment* of the line.

If now we discuss all possible systems of form (126), subject merely to the restriction that m, n, and a shall not, all three, be equal to zero, but dropping all other restrictions as to the vanishing of these quantities, we obtain again congruences of the six types of Article 9. Consequently these congruences may be regarded as arising from properties Ia, Ib, II, and III by a mere modification of property Ia, namely by permitting the focal sheets to be ruled.

The congruences which correspond to a system of form (1) with constant coefficients, such as (126), have been studied before, without imposing any restrictions as to the reality of the coefficients. If $m \neq 0$, $n \neq 0$, every congruence corresponding to such a system has the following properties:

it belongs to a linear complex, all of the congruences derived from it by Laplace transformations also belong to linear complexes and are, besides, projectively equivalent to each other. These properties are, moreover, characteristic of such systems, and imply that the focal sheets are quadrics which have a skew quadrilateral in common.* Clearly, in the case of type A, this quadrilateral is composed of four imaginary generators of the Riemann sphere.

The University of Chicago,
February 29, 1920.

* *Brussels Paper*, p. 77.

ON THE EQUILIBRIUM OF A FLUID MASS AT REST*

BY

JAMES W. ALEXANDER

1. The following question was brought up a considerable time ago by both Liapounoff† and Poincaré,‡ but has apparently not been answered up to the present:

"Consider a homogeneous incompressible fluid whose particles attract one another according to Newton's law and which is acted on by no external forces. Then, are there any positions of equilibrium for the fluid besides the sphere?"

It will be shown that there are no such positions, whether of stable or unstable equilibrium.

2. A necessary condition for equilibrium will be obtained by examining an approximating figure made up of elementary parallelepipeds, or parallel rods. The rods will be treated as rigid and free to move in the direction of their lengths only, so that perpendicular distances between them remain unchanged. They will be so chosen that whenever two collinear rods are moved into contact with one another their ends will fit together exactly and the rods will become merged into one.

If the approximating system consists of two rods only, it can be seen by inspection that its potential energy diminishes continuously as the centers of the rods approach one another. Equilibrium can, therefore, only occur if the rods are touching end to end or if they are symmetrical about a perpendicular line through their centers.

If there are more than two rods, the potential energy of the approximating system is equal to the sum of the potential energies of all sub-systems consisting of two rods only. Suppose the rods are set in motion in such a way that the center of each rod approaches a fixed perpendicular plane, π, with a velocity equal to its instantaneous absolute distance from π. Then, as the system moves, the distance between the centers of two rods never increases, while, on the contrary, it decreases whenever the centers are at unequal distances from π.

* Presented to the Society, February 28, 1920.

† Liapounoff, *Sur le corps du potentiel maximum,* Communications de la Société Mathématique de Kharkow (1887), pp. 63–73.

‡ Poincaré, *Sur un théorème de M. Liapounoff relatif à l'équilibre d'une masse fluide,* Comptes Rendus, vol. 104 (1887), p. 622.

The rate-loss of potential energy, $-dW/dt$, changes abruptly whenever two collinear rods merge into one, since the rods undergo a sudden change of velocity at that instant. The value of $-dW/dt$ stays greater than zero, however, so long as two rods are collinear without touching and so long as the system is not symmetrical about a plane through its center of mass and parallel to π. Under either of these conditions, therefore, the approximating system can not be in equilibrium.

3. The problem for the fluid mass itself can now be solved by an obvious passage to the limit if we impose upon the boundary the restrictions that are usually assumed in a discussion of this kind. As we wish to handle the perfectly general case, however, we merely observe at this point that the limiting process is obvious provided the fluid is convex. Therefore, since the orientation of the rods of the approximating systems does not affect the argument in any way, we have the

THEOREM. *A convex figure of equilibrium is symmetrical about every plane through its center of mass and is therefore bounded by a sphere.*

It remains to be shown that a figure which is not convex cannot be one of equilibrium.

4. Although a perfectly general boundary will be permitted, spurious "boundaries" either wholly within or wholly without the fluid are to be excluded. We therefore assume: that every interior point is within a sphere which encloses interior points only; that every exterior point is within a sphere which encloses exterior points only; that within every sphere about a boundary point there are both interior and exterior points. We shall also assume that there is an upper bound, L, to the distance between two interior points.

Under the above assumptions, we must be prepared to meet the case where the boundary of the fluid is a set of points of measure greater than zero, so that the volume depends on whether we decide to count in or leave out the boundary points.* Let us agree to define the volume, T, as the measure of the interior points only. The potential energy will then be the Lebesgue integral

$$W = -\frac{k}{2}\int\int \frac{d\tau d\tau'}{R(\tau,\tau')}$$

extended over the interior points, where k is the gravitational constant and $R(\tau,\tau')$, the ultimate distance between the elements $d\tau$ and $d\tau'$. If space be cut up into cubes by means of three systems of parallel planes, the sum of the volumes of the cubes that are wholly within the fluid approaches the limit T as the distance between parallel planes approaches zero. Moreover,

* For the analogous case in two dimensions, cf. Osgood, *A Jordan curve of positive area*, these Transactions, vol. 4 (1903), pp. 107–112.

the potential energy of the system composed of interior cubes approaches the limit W.

5. Since the fluid is not convex, a segment PQP' can always be found such that P and P' are both within the fluid while Q is without it or on the boundary. Moreover, we can always arrange to make Q an outside point by displacing the segment PQP' a bit, if necessary.

Let us approximate the fluid figure, which we shall call F, by a sequence of figures, $F_1, F_2, F_3, \cdots$ made up of rods parallel to the segment PQP', such that P and P' are interior points of F_1 and that every figure F_i is contained within all subsequent figures of the series and in F. Then, to be sure, a plane π perpendicular to PQP' can be chosen with reference to which the figures F_i can be set in motion in the manner described in § 2. Furthermore a perfectly unambiguous limiting motion can be determined by allowing i to increase indefinitely. Unfortunately, the limiting motion so changes the internal structure of the fluid by transforming interior points into boundary points and vice versa, that we cannot be sure among other things that the volume remains constant. We therefore proceed in a more roundabout way.

About the points Q, P, and P' we draw spheres of radii l, $l/3$, and $l/3$ respectively, where l is so small that the sphere about Q is exterior to the fluid F, while the spheres about P and P' are interior to the first approximating figure F_1. We can then obtain a lower bound for the rate-loss of potential energy of the figure F_i when set in motion in the way described above, such that this lower bound depends on l and L (§ 4) alone and not on i.

Let us choose the plane π perpendicular to PQP' and through the center of mass of F. Then consider what happens as the motion begins. The spheres about P and P' have between them a gap of exterior points interior to the sphere about Q and of width at least l. Consequently, they approach one another with a velocity greater than l. Moreover their mutual attraction is at least $k(T^2/L^2)$, where T is the volume of each.

Therefore, if the potential energy of F_i be W_i, we have at the start of a motion

$$-\frac{dW_i}{dt} > \phi, \qquad \text{where} \qquad \phi = k\frac{T^2}{L^2}l,$$

while during an interval of time $\bar{t}$ depending on l and L but not on i, we have

$$-\frac{dW_i}{dt} > \phi/2, \qquad \bar{t} > t > 0. \tag{1}$$

Suppose now, we expand the figures F_i to the volume of F by similarity transformations which preserve directions and leave fixed the center of mass of F. We then obtain a second sequence of approximating figures F_i' of the

same volume as F. Moreover, when the figures F'_i are set in notion in the same way as the figures F_i, we have from (1)

$$(2) \qquad -\frac{dW'_i}{dt} > \phi/2, \qquad \bar{t} > t > 0$$

a fortiori.

Let us denote by ϵ'_i the maximum displacement undergone by any point of the figure F'_i after a motion lasting a time t. Then we have

$$Ldt \geqq d\epsilon'_i,$$

and therefore,

$$-\frac{dW'_i}{d\epsilon_i} > \frac{\phi}{2L}, \qquad \bar{t} > t > 0.$$

Furthermore, if $\Delta W'_i$ be the total change in potential energy of the figure F'_i after motion lasting a time t and if $\Delta s'_i$ be the maximum distance from a point on the interior or boundary of the displaced figure to the nearest point on the interior or boundary of the figure in its original position, we have

$$\Delta W'_i = \int_0^t dW, \qquad \text{and} \qquad \Delta s'_i \leqq \int_0^t d\epsilon'_i.$$

Therefore

$$(3) \qquad -\frac{\Delta W'_i}{\Delta s'_i} > \frac{\phi}{2L}.$$

Now, let $\Delta \overline{W}_i$ be the difference between the potential energy of the displaced figure F'_i at a time t and the potential energy of the fluid F, and let $\Delta \bar{s}_i$ be the maximum distance from a point on the interior or boundary of the displaced F_i and the nearest point on the interior or boundary of F. Then, if we hold t fixed, we have

$$(4) \qquad \lim_{i=\infty}\left[\frac{\Delta W'_i}{\Delta s'_i} - \frac{\Delta \overline{W}_i}{\Delta \bar{s}_i}\right] = 0,$$

since the undisplaced figure F_i tends towards the figure F in the limit. Therefore by (3) and (4), we can find, corresponding to any t, a value of i such that

$$(5) \qquad -\frac{\Delta \overline{W}_i}{\Delta \bar{s}_i} > \frac{\phi}{2L}.$$

Finally, let t run through a decreasing sequence of values with zero as limit. Then, corresponding to each value of t, we can find a displaced figure F_i such that the relation (5) holds. These displaced figures evidently approach F as limit, since i increases and the time of displacement decreases towards zero. Therefore, in view of (5), the figure F is not in equilibrium. Hence:

The sphere is the only figure of equilibrium.

6. Before leaving this matter, let us say one word about the more general problem, where the fluid mass is rotating about an axis through its center of gravity. The methods used in this paper show us at once that the fluid cannot be in equilibrium unless symmetrical about a plane perpendicular to the axis of rotation and unless a line parallel to this axis cuts the fluid in at most one segment.

New York,
January, 1919.

CONCERNING APPROACHABILITY OF SIMPLE CLOSED AND OPEN CURVES*

BY

JOHN ROBERT KLINE

Schoenflies† was the first to formulate the converse of the fundamental theorem of C. Jordan‡ that a simple closed curve§ lying wholly within a plane decomposes the plane into an inside and an outside region. The statement of this converse theorem is as follows: *Suppose K is a closed, bounded set of points lying in a plane S and that $S - K = S_1 + S_2$, where S_1 and S_2 are point-sets such that* (1) *every two points of S_i ($i = 1, 2$) can be joined by an arc lying entirely in S_i* (2) *every arc joining a point of S_1 to a point of S_2 contains at least one point of K* (3) *if O is a point of K and P is a point not belonging to K, then P can be joined to O by an arc that has no point except O in common with K. Every point-set that satisfies these conditions is a simple closed curve.* Schoenflies used metrical hypotheses in his proof. Lennes gave a proof of this converse theorem using straight lines.|| R. L. Moore pointed out that a proof similar in large part to that of Lennes can be carried through with the use of arcs and closed curves on the basis of his system of postulates Σ_3, thus furnishing a non-metrical proof of the converse theorem.¶

In all these proofs the condition numbered three, the condition of approachability (erreichbarkeit) plays a fundamental rôle. It is the purpose of the present paper to study the effect of substituting for the condition of approach-

* Presented to the Society, April, 1918.

† Cf. A. Schoenflies, *Ueber einen grundlegenden Satz der Analysis Situs*, Nachrichten der Göttinger Gesellschaft der Wissenschaften, 1902, p. 185.

‡ C. Jordan, *Cours d'Analyse*, 2d ed., Paris, 1893, p. 92.

§ If A and B are distinct points, then a *simple continuous arc* from A to B is defined by Lennes as a bounded, closed, connected set of points containing A and B, but containing no proper connected subset containing both A and B, Cf. N. J. Lennes *Curves in non-metrical analysis situs with an application in the calculus of variations*, American Journal of Mathematics, vol. 33 (1911), p. 308. A *simple closed curve* is a set of points composed of two arcs AXB and AYB having no point in common other than A and B. Hereafter in this paper "arc" and "closed curve" will be considered synonymous with "simple continuous arc" and "simple closed curve," respectively.

|| Cf. N. J. Lennes, loc. cit., § 5.

¶ Cf. R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), p. 59.

ability, the condition that the set is "*connected in kleinem*."* The results obtained are embodied in the following theorem:

THEOREM *A*. *Suppose K is a closed plane point-set, S is the set of all points of the plane, while $S - K = S_1 + S_2$, where S_1 and S_2 are two mutually exclusive domains*† *such that every point of K is a common boundary point of S_1 and S_2. Then a necessary and sufficient condition that K be either a simple closed curve or an open curve*‡ *is that K be connected in kleinem.*

That the condition stated in Theorem *A* is necessary is evident. I will proceed to show that it is sufficient. Suppose K is a connected in kleinem set satisfying the conditions stipulated in Theorem *A*. Then the following lemmas hold true:

LEMMA *A*. *Every arc joining a point of S_1 to a point of S_2 contains a point of K.*

Proof. Suppose it were possible to draw an arc from a point P_1 of S_1 to a point P_2 of S_2 that contains no point of K. Then let us divide the arc $P_1 P_2$ into two sets, M_1 and M_2, where M_1 is the set of all points of $P_1 P_2$ that belong to S_1, while M_2 is the set of all points of $P_1 P_2$ which belong to S_2. As $P_1 P_2$ is a connected point-set either M_1 contains a limit point of M_2 or M_2 contains a limit point of M_1.

Case I. A point F of M_1 is a limit point of M_2. As F is a point of the domain S_1, there exists a region containing F and lying entirely in S_1. As S_1 and S_2 are mutually exclusive domains, this region contains no point of M_2. Hence F cannot be a limit point of M_2.

Case II. A point G of M_2 is a limit point of M_1. This is impossible as in Case I.

Hence we are led to a contradiction if we suppose our lemma false.

LEMMA *B*. *The set K is connected.*

Proof. Suppose K were not connected. Then it could be divided into two mutually exclusive sets K_1 and K_2, neither of which contains a limit point of the other one. Let P_i $(i = 1, 2)$ denote a point of K_i. Put about P_i a circle R_i having P_i as center and such that R_i and its interior lie entirely

* Cf. Hans Hahn, *Ueber die allgemeinste ebene Punktmenge die stetiges Bild einer Strecke ist*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 23 (1914), pp. 318–22. According to Hahn, a set of points C is said to be *connected in kleinem* if, whenever P is a point of C, ϵ a positive number and K a circle of radius $1/\epsilon$ with center at P, then there exists within K and with center at P, another circle $K_{\epsilon, P}$ such that if X is a point of C within $K_{\epsilon, P}$ then X and P lie together in some connected subset of C that lies entirely within K.

† A *domain* is a connected set of points M such that if P is a point of M, then there is a region that contains P and lies in M.

‡ An *open curve* is defined by R. L. Moore as a closed, connected, set of points M such that if P is a point of M, then $M - P$ is the sum of two mutually exclusive connected point-sets, neither of which contains a limit point of the other.

without R_{i+1}.* As K is connected in kleinem, there exists a circle $\bar{R}$, lying within R_i and with center at P_i such that if X_i is a point of K within $\bar{R}_i$, then X_i and P_i lie on some connected subset of K lying within R_i. It may easily be shown that X_i can be joined to P_i by a simple continuous arc of K lying entirely within R_i.† As every point of K is a common boundary point of S_1 and S_2, then there exists within $\bar{R}_i$ a point M_{ij} $(j = 1, 2)$ belonging to S_j. As S_j is a domain, then there exists a simple continuous arc $M_{1j} K_j M_{2j}$ lying entirely in S_j. Join M_{ij} to P_i by a simple continuous arc $M_{ij} L_{ij} P_i$ lying entirely within $\bar{R}_i$ and let G_{ij} denote the first point of K on the arc $M_{ij} L_{ij} P_i$ following M_{ij}. Then we may join G_{i1} to G_{i2} by an arc $G_{i1} F_i G_{i2}$ belonging to K and lying entirely within R_i. The point-set $G_{11} M_{11}$ (on $M_{11} L_{11} P_1$) $+ M_{11} K_1 M_{21} + M_{21} G_{21}$ (on $M_{21} L_{21} P_2$) contains as a subset a simple continuous arc $G_{11} H_1 G_{21}$ lying except for its endpoints entirely in S_1, while the set $G_{12} M_{12}$ (on $M_{12} L_{12} P_1$) $+ M_{12} K_2 M_{22} + M_{22} G_{22}$ (on $M_{22} L_{22} P_2$) contains as a subset a simple continuous arc $G_{12} H_2 G_{22}$ lying except for its endpoints entirely in S_2. We then have a closed curve $G_{11} F_1 G_{12} H_2 G_{22} - F_2 G_{21} H_1 G_{11}$ such that the arcs $G_{11} F_1 G_{12}$ and $G_{21} F_2 G_{22}$ lie entirely on K and within R_1 and R_2, respectively, while $\underline{G_{11} H_1 G_{21}}$‡ and $\underline{G_{12} H_2 G_{22}}$ belong to S_1 and S_2, respectively.

All points of $G_{11} F_1 G_{12}$ belong to K_1. For suppose a point H of $G_{11} F_1 G_{12}$ belonged to K_2. As H is joined to G_{11}, which in turn can be joined to P_1 by an arc of K lying entirely within R_1, it follows that H can be joined to P_1 by an arc HFP_1 of K lying entirely within R_1. Let $[\bar{H}_1]$ denote the set of all points of HFP_1 belonging to K_1 while $[\bar{H}_2]$ denotes the set of all points of HFP_1 belonging to K_2. Clearly neither of these sets contains a limit point of the other. Hence the arc HFP_1 is not a connected point-set. Hence the supposition that H belongs to K_2 has led to a contradiction. In like manner, all points of $G_{21} F_2 G_{22}$ belong to K_2.

The interior of $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$ must contain at least one point of K. For suppose it does not contain a point of K. Then the interior of $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$ is a subset of $S_1 + S_2$. Suppose it contains a point H of S_1. Then H can be joined to H_2 by an arc HXH_2 lying except for H_2 entirely within $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$.§ Let $[W_1]$ denote the set of all points of HXH_2 belonging to S_1 while $[W_2]$ denotes the set of all points of HXH_2 which are points of S_2. Clearly neither of these sets contains

* It is understood that subscripts are reduced modulo 2,

† Cf. R. L. Moore, *A theorem concerning continuous curves*, Bulletin of the American Mathematical Society, vol. 23 (1917). While Professor Moore's theorem states that every two points of a continuous curve can be joined by a simple continuous arc lying entirely on the given continuous curve, it is clear that his methods suffice to prove the above stronger statement.

‡ If AXB is an arc, then the symbol $\underline{AXB}$ will denote $AXB - A - B$,

§ Cf. R. L. Moore, *Foundations of plane analysis situs*, loc. cit., Theorem 39, pp. 153–5.

a limit point of the other. Hence the arc HXH_2 is not a connected point-set. In like manner the supposition that there is within $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$, a point of S_2 leads to a contradiction. Hence $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$ must enclose a point of K.

Let $[V_2]$ denote the set of all points V_2 such that either (1) V_2 is a point of $G_{21} F_2 G_{22}$, or (2) V_2 is a point such that there exists a closed connected set $V_2 XF'_2$ belonging to K and lying within or on $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$ and such that F'_2 is a point of $G_{21} F_2 G_{22}$. As K is connected in kleinem it may easily be proved that $[V_2]$ is a closed set. It is also true that all points of $[V_2]$ belong to K_2. Hence no point of $G_{11} F_1 G_{12}$ either belongs to or is a limit point of $[V_2]$. It may also be proved with the use of the in kleinem property that no point of $[V_2]$ is a limit point of a set of points of K lying within $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$ and containing no point of V_2. There exists an arc $H_1 YH_2$ such that (1) $H_1 YH_2$ is a subset of the interior of $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$ and (2) $H_1 YH_2$ contains no points of $[V_2]$.* Let $[V_1]$ denote the set of all points of K within or on the closed curve, $H_1 YH_2 G_{22} F_2 G_{21} H_1$, not belonging to $[V_2]$. The set $[V_1]$ is closed. Put about each point of $[V_1]$ a circle lying entirely within $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$ and containing within it or on its boundary no point of $[V_2]$. By the Heine-Borel Property, there exists a finite number of circles of the above set, $C_1, C_2, \cdots, C_n$, covering $[V_1]$. With the use of Theorems 41, 42, 43, and 44 of Professor Moore's Foundations we may easily obtain from the set $C_1, C_2, \cdots, C_n$ and the closed curve $G_{11} F_1 G_{12} H_2 YH_1 G_{11}$, a new closed curve $G_{11} F_1 G_{12} H_2 ZH_1 G_{11}$, where the arc $H_1 G_{11} F_1 G_{12} H_2$ of the new closed curve $G_{11} F_1 G_{12} H_2 ZH_1 G_{11}$ is the arc $H_1 G_{11} F_1 G_{12} H_2$ of $G_{11} F_1 G_{12} H_2 YH_1 G_{11}$ and where $H_2 ZH_1$ is free from points of K and lies within $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$. But then we have a point of S_1 joined to a point of S_2 by an arc containing no point of K. Thus the supposition that K is not connected, leads to a contradiction.

LEMMA C. *If K contains one simple closed curve J, then all points of K belong to J.*

Proof. Suppose Lemma C is not true. Then K contains a closed curve J and at least one point P not on J. Two cases may arise:

Case I. P is within J. As every point of K is a common boundary point of S_1 and S_2, the interior of J contains a point P_1 of S_1 and a point P_2 of S_2. The exterior of J cannot contain a point $\overline{P}_1$ of S_1. For suppose it did. Then any arc from P_1 to $\overline{P}_1$ would contain a point of J and hence a point of K, contrary to the fact that S_1 is a domain. In like manner no point of S_2 can be in the exterior of J. Hence the exterior of J must be a subset of K, while

* Cf. my paper, *A definition of sense on plane closed curves in non-metrical analysis situs*, Annals of Mathematics, vol. XIX (1918), Theorem D, pp. 188–9.

S_1 and S_2 are subsets of the interior of J. But this is impossible because no point without J is a limit point of a set of points lying entirely within J thus making it impossible that every point of K be a common boundary point of S_1 and S_2. Hence the supposition that P is within J has led to a contradiction.

Case II. P is without J. Case II may be proved impossible by an argument similar to that used in Case I.

An immediate consequence of Lemma C is that if K is not a simple closed curve, then there is but one K-arc from a point A of K to a distinct point B of K.

LEMMA D. *The set K does not contain three arcs OP_1, OP_2, and OP_3, no two of which have a common point other than O.*

Proof. Suppose Lemma D were false. Then there would exist three arcs OP_1, OP_2, and OP_3, no two of which have a point in common other than O. Put about P_i $(i = 1, 2, 3)$ a circle C_i such that the point-set OP_{i+1}* $+ OP_{i+2}$ is a subset of the exterior of C_i and such that C_i has no point in common with $C_{i+1} + C_{i+2}$. As K is connected in kleinem, there exists within C_i and with center at P_i, another circle C_{P_i, C_i} such that if X_i is a point of K within C_{P_i, C_i}, then there is an arc from X_i to P_i every point of which is a point of K and which lies entirely within C_i.† As all points of K are limit points of both S_1 and S_2, C_{P_i, C_i} must contain at least one point $P_{i,1}$ of S_1. As S_1 is a domain, there is an arc $P_{11} P_{21}$ from P_{11} to P_{21} all points of which belong to S_1. Join $P_{i,1}$ to P_i by an arc $P_{i,1} P_i$ lying entirely within C_{P_i, C_i} and let X_i denote the first point of the arc $P_{i,1} P_i$ after $P_{i,1}$, which belongs to K. There exists an arc $X_i P_i$ from X_i to P_i belonging to K and lying entirely within C_i. Let P_i' denote the first point of the arc $X_i P_i$ which is on OP_i. The point-set $P_1' X_1 + X_1 P_{11} + P_{11} P_{21} + P_{21} X_2 + X_2 P_2'$ contains as a subset an arc $P_1' F_1 P_2'$ such that (1) $P_1' F_1 P_2'$ has no point in common with $OP_1 + OP_2 + OP_3$, (2) all points of $P_1' F_1 P_2'$ belong to either K or S_1, (3) at least one point, F_1, of S_1 is a point of $P_1' F_1 P_2'$. By methods similar to those just employed, we may construct an arc $Q_1' H_2 Q_2'$ from a point Q_1' of OP_1 to a point Q_2' of OP_2' such that (1) Q_i' is on OP_i between P_i' and O, (2) all points of $Q_1' H_2 Q_2'$ belong to either S_2 or K, (3) except for Q_1' and Q_2', $Q_1' H_2 Q_2'$ has no point in common with $P_1' FP_2' + OP_1 + OP_2 + OP_3$, (4) at least one point H_2 of $Q_1' H_2 Q_2'$ belongs to S_2. Two cases may arise:

Case I. $Q_1' H_2 Q_2'$ is entirely within $OP_1' F_1 P_2' O$. Then the interior of $OP_1' F_1 P_2' O = Q_1' H_2 Q_2' +$ the interior of $OQ_1' H_2 Q_2' O +$ the interior of $P_1' F_1 P_2' Q_2' H_2 Q_1' P_1'$. The point-set $OP_3 + P_3$ is either entirely within or entirely without $OQ_1' H_2 Q_2' O$.

(a) Suppose $OP_3 + P_3$ is entirely within $OQ_1' H_2 Q_2' O$. Then $OQ_1' H_2 Q_2' O$

* It is understood throughout this argument that subscripts are reduced modulo 3.

† See an earlier footnote.

must enclose at least one point L of S_1. But then an arc from L to F_1 must contain at least one point of $OQ_1'H_2Q_2'O$. Hence, as $OQ_1'H_2Q_2'O$ is a subset of $K+S_2$, no such arc LF_1 can lie entirely in S_1, contrary to the fact that S_1 is a domain.

(*b*) Suppose OP_3+P_3 is entirely without $OQ_1'H_2Q_2'O$. It follows that OP_3+P_3 is entirely without $OP_1'F_1P_2'O$. Then the exterior of $OP_1'F_1P_2'O$ contains at least one point M of S_2. Then any arc from M to H_2 must contain at least one point of $OP_1'F_1P_2'O$ and hence at least one point not in S_2. But this is contrary to the fact that S_2 is a domain.

Thus in Case I we are led to a contradiction.

Case II. $\overline{Q_1'H_2Q_2'}$ is without $OP_1'F_1P_2'O$. We may show that Case II is impossible by methods similar to those used in Case I.

LEMMA *E*. *If O is a point of K and P is a point of S_i $(i=1,2)$ then there exists at least one arc OP such that $\overline{OP}+P$ is a subset of S_i.*

Proof. Two conceivable cases may arise.

Case I. There exist points A_1 and A_2 of $K[A_1\neq O\neq A_2]$ such that O is a point of the arc A_1OA_2 belonging to K. By the same methods as were used in the preceding lemma we may construct an arc $A_1'F_1A_2'$ such that (1) on A_1OA_2 the order $A_1A_1'OA_2'A_2$ holds, (2) $A_1'F_1A_2'$ is a subset of S_1+K, (3) at least one point F_1 of $A_1'F_1A_2'$ is a point of S_1, (4) no point of $\overline{A_1'FA_2'}$ belongs to A_1OA_2. The point O is not a limit point of $K-A_1'OA_2'$. For suppose it were. Then it would be a sequential limit point of a set of points $P_1, P_2, \cdots$, every one of which belongs to $K-A_1'OA_2'$. Put about O as center a circle M such that A_1' and A_2' are both without M. As K is connected in kleinem there exists another circle $\overline{M}$ lying within M and having its center at O such that if X is a point of K within $\overline{M}$, then X and O can be joined by an arc of K lying entirely within M. Let $\overline{P}$ denote that point of the set $P_1, P_2, \cdots$ of lowest subscript which lies within $\overline{M}$, while $\overline{P}O$ denotes an arc of K from $\overline{P}$ to O lying entirely within M. Let O' denote the first point of $\overline{P}O$ which is on $A_1'OA_2'$. Then the set K contains three arcs $A_1'O'$, $A_2'O'$, and PO', no two of which have a point in common other than O'. But this is contrary to Lemma D. Hence O cannot be a limit point of $K-A_1'OA_2'$. There exists a closed curve G enclosing O but enclosing no points of $A_1'FA_2'+[K-A_1'OA_2']$. Then there exist two closed curves J_1' and J_2' such that (1) every point of J_1' or J_2' belongs either to G or to $A_1'F_1A_2'OA_1'$ (2) O is on J_1' and on J_2' (3) every point within J_1' is within $A_1'F_1A_2'OA_1'$ while every point within J_2' is without $A_1'F_1A_2'OA_1'$ (4) every point within either J_1' or J_2' is within G.* It is clear that either the interior of J_1' or the interior of J_2' is a subset of S_1 while the interior of the other of these two closed curves is a subset of S_2. Let J_1 denote that one whose interior is a subset of S_1 while

* Cf. R. L. Moore, *Foundations*, Theorem 43, pp. 156–7.

J_2 denotes the one whose interior is a subset of S_2. Let E denote a point within J_1, while P_1 is any other point of S_1. There exists an arc EO such that $EO - O$ is a subset of the interior of J_1.* As S_1 is a domain, there is an arc EP_1 lying entirely in S_1. The point-set $EO + EP_1$ contains as a subset an arc from P_1 to O lying except for O entirely in S_1. In like manner we may show that any point P_2 of S_2 can be joined to O by an arc lying except for O entirely in S_2.

Case II. There do not exist two distinct points A_1 and A_2 of K such that O is on an arc of K from A_1 to A_2. Let A denote a point of K different from O while ARO denotes an arc of K from A to O. By an argument similar to that employed in Case I we may show that if O were a limit point of $K - ARO$, then either there would exist three arcs AR', $R'O$, and $R'P$, no two of which have a point in common other than R' or there would exist a point A', $(A \neq A' \neq O)$ such that O is an arc of K from A' to A. But the first of these possibilities contradicts Lemma D while the second is contrary to the hypothesis of Case II. Hence O cannot be a limit point of $K - ARO$. Put about O a circle C that neither contains or encloses any point of $K - ARO$. Let $P_1, P_2, \cdots$ denote a set of points of S_1 approaching O as their sequential limit point. It is possible to pass at least one simple continuous arc† through $ARO + P_1 + P_2 + \cdots$. Let $P_1'ORA$ denote one such arc. If the interval OP_1' of the arc $P_1'ORA$ does not lie entirely with in C, let P' denote the first point which it has in common with C. Otherwise let P' denote P_1'. Let $\overline{P}$ denote that point of the set $P_1, P_2, \cdots$ of lowest subscript lying on OP'. It is clear that the sub-arc $O\overline{P}$ of $P_1'ORA$ lies, except for O, entirely in S_1. Let F_1 denote any other point of S_1. Join F_1 to $\overline{P}$ by an arc lying entirely in S_1. Then the point-set $O\overline{P} + \overline{P}F_1$ contains as a subset an arc from O to F_1, lying except for O entirely in S_1.

In like manner we may show that if F_2 is a point of S_2, F_2 can be joined to O by an arc lying except for O entirely in S_2.

LEMMA F. *A necessary and sufficient condition that K be bounded, is that either S_1 or S_2 be bounded.*

Proof. The condition is necessary. Let us suppose that K is bounded while neither S_1 nor S_2 is bounded. As K is bounded, there is a circle C such that all points of K are within C. As S_1 and S_2 are unbounded, there is a point P_1 of S_1 and a point P_2 of S_2 without C. Join P_1 and P_2 by an arc lying entirely without C. By Lemma A, this arc must contain a point of K. But all points of K are within C. Hence we are led to a contradiction if we suppose our condition is not necessary.

* Cf. R. L. Moore, *Foundations*, Theorem 39, pp. 153–5.

† Cf. R. L. Moore and J. R. Kline, *On the most general closed point-set through which it is possible to pass a simple continuous arc*, Annals of Mathematics, vol. XX (1919), pp. 218–23.

The condition is sufficient. For suppose S_1 is bounded while K is unbounded. Since S_1 is bounded, there exists a circle C enclosing S_1. Since K is unbounded, it contains a point P without C. The point P cannot be a limit point of S_1. But this is contrary to hypothesis.

Proof of Theorem A. Two cases may arise:

Case I. K is bounded. Then, by Schoenflies' Theorem and the preceding lemmas, it follows that K is a simple closed curve.

Case II. K is unbounded. It follows, by Lemma F that neither S_1 nor S_2 is bounded. Then K is an open curve. For a proof of this statement see my paper, "The converse of the theorem concerning the division of a plane by an open curve."*

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* Cf. these Transactions, vol. 18 (1917), pp. 177–184.